The Dimension Spectrum Conjecture for Planar Lines

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Abstract

Let $L_{a,b}$ be a line in the Euclidean plane with slope a and intercept b. The dimension spectrum $\mathrm{sp}(L_{a,b})$ is the set of all effective dimensions of individual points on $L_{a,b}$. The dimension spectrum conjecture states that, for every line $L_{a,b}$, the spectrum of $L_{a,b}$ contains a unit interval.

In this paper we prove that the dimension spectrum conjecture is true. Let (a, b) be a slope-intercept pair, and let $d = \min\{\dim(a, b), 1\}$. For every $s \in (0, 1)$, we construct a point x such that $\dim(x, ax + b) = d + s$. Thus, we show that $\mathrm{sp}(L_{a,b})$ contains the interval $(d, 1 + d)$. Results of Turetsky [16], and Lutz and Stull [11], show that $\mathrm{sp}(L_{a,b})$ contain the endpoints d and $1 + d$. Taken together,

$$[d, 1 + d] \subseteq \mathrm{sp}(L_{a,b}),$$

for every planar line $L_{a,b}$.

1 Introduction

The effective dimension, $\dim(x)$, of a Euclidean point $x \in \mathbb{R}^n$ gives a fine-grained measure of the algorithmic randomness of x. Moreover, due to its strong connection to (classical) Hausdorff dimension, effective dimension has proven to be geometrically meaningful[4, 13, 2, 10]. A natural question, therefore, is the behavior of the effective dimensions of points on a given line.

Let $L_{a,b}$ be a line in the Euclidean plane with slope a and intercept b. Given the point-wise nature of effective dimension, one can study the *dimension spectrum* of $L_{a,b}$. That is, the set

$$\mathrm{sp}(L_{a,b}) = \{\dim(x, ax + b) \,|\, x \in \mathbb{R}\}$$

of all effective dimensions of points on $L_{a,b}$. Jack Lutz posed the *dimension spectrum conjecture* for lines. He conjectured that the dimension spectrum of every line in the plane contains a unit interval.

The first progress on this conjecture was made by Turetsky. He showed [16] that $1 \in \mathrm{sp}(L_{a,b})$ for every line $L_{a,b}$. In [11], Lutz and Stull showed that the effective dimension of points on a line is intimately connected to problems in fractal geometry (more on this below). Among other things, they proved that $1 + d \in \mathrm{sp}(L_{a,b})$ for every line $L_{a,b}$, where $d = \min\{\dim(a, b), 1\}$.

In this paper, we prove that dimension spectrum conjecture for planar lines is true. For every $s \in (0, 1)$, we construct a point x such that $\dim(x, ax+b) = d+s$, where $d = \min\{\dim(a, b), 1\}$. This, combined with the results of Turetsky and Lutz and Stull, prove that

$$[d, 1 + d] \subseteq \mathrm{sp}(L_{a,b}),$$

for every planar line $L_{a,b}$.

Apart from its intrinsic interest, recent work has shown that the effective dimensions of points has connections to deep problems in classical analysis [8, 10, 11, 15]. Lutz and Lutz [7] proved the point-to-set principle, which characterizes the Hausdorff dimension of a *set* by effective dimension of its *individual points*. Lutz and Stull [11], using the point-to-set principle, showed that lower bounds on $\dim(x, ax + b)$ implies lower bounds on the Hausdorff dimension of Furstenberg sets.

2 Preliminaries

The *conditional Kolmogorov complexity* of a binary string $\sigma \in \{0,1\}^*$ given binary string $\tau \in \{0,1\}^*$ is

$$K(\sigma|\tau) = \min_{\pi\in\{0,1\}^*} \{\ell(\pi) : U(\pi, \tau) = \sigma\} ,$$

where U is a fixed universal prefix-free Turing machine and $\ell(\pi)$ is the length of π. The *Kolmogorov complexity* of σ is $K(\sigma) = K(\sigma|\lambda)$, where λ is the empty string. We stress that the choice of universal machine effects the Kolmogorov complexity by at most an additive constant (which, especially for our purposes, can be safely ignored). See [5, 14, 3] for a more comprehensive overview of Kolmogorov complexity.

We can extend these definitions to Euclidean spaces by introducing "precision" parameters [9, 7]. Let $x \in \mathbb{R}^m$, and $r, s \in \mathbb{N}$. The *Kolmogorov complexity of x at precision r* is

$$K_r(x) = \min \{K(p) : p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m\} .$$

The *conditional Kolmogorov complexity of x at precision r given $q \in \mathbb{Q}^m$* is

$$\hat{K}_r(x|q) = \min \{K(p) : p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m\} .$$

The *conditional Kolmogorov complexity of x at precision r given $y \in \mathbb{R}^n$ at precision s* is

$$K_{r,s}(x|y) = \max \left\{\hat{K}_r(x|q) : q \in B_{2^{-r}}(y) \cap \mathbb{Q}^n\right\} .$$

We abbreviate $K_{r,r}(x|y)$ by $K_r(x|y)$.

The *effective Hausdorff dimension* and *effective packing dimension*[1] of a point $x \in \mathbb{R}^n$ are

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r} \quad \text{and} \quad \mathrm{Dim}(x) = \limsup_{r\to\infty} \frac{K_r(x)}{r}\,.$$

Intuitively, these dimensions measure the density of algorithmic information in the point x.

By letting the underlying fixed prefix-free Turing machine U be a universal *oracle* machine, we may *relativize* the definition in this section to an arbitrary oracle set $A \subseteq \mathbb{N}$. The definitions of $K^A_r(x)$, $\dim^A(x)$, $\mathrm{Dim}^A(x)$, etc. are then all identical to their unrelativized versions, except that U is given oracle access to A. Note that taking oracles as subsets of the naturals is quite general. We can, and frequently do, encode a point y into an oracle, and consider the complexity of a point *relative* to y. In these cases, we typically forgo explicitly referring to this encoding, and write e.g. $K^y_r(x)$.

Among the most used results in algorithmic information theory is the *symmetry of information.* In Euclidean spaces, this was first proved, in a slightly weaker form in [7], and in the form presented below in [11].

Lemma 1. *For every $m, n \in \mathbb{N}$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r, s \in \mathbb{N}$ with $r \geq s$,*

(i) $\left|K_r(x|y) + K_r(y) - K_r(x,y)\right| \leq O(\log r) + O(\log\log \|y\|)\,.$

(ii) $|K_{r,s}(x|x) + K_s(x) - K_r(x)| \leq O(\log r) + O(\log\log \|x\|)\,.$

2.1 Dimensions of points on lines

The proof of our main theorem will use the tools and techniques introduced by Lutz and Stull [11]. The first lemma intuitively states the following. Suppose that $L_{a,b}$ is the only line of low complexity which intersects $(x, ax + b)$. Then it is possible to compute (a, b) given $(x, ax + b)$ by simply enumerating over all low complexity lines.

Lemma 2 (Lutz and Stull [11])**.** *Suppose that $a, b, x \in \mathbb{R}$, $m, r \in \mathbb{N}$, $\delta \in \mathbb{R}_+$, and $\varepsilon, \eta \in \mathbb{Q}_+$ satisfy $r \geq \log(2|a| + |x| + 5) + 1$ and the following conditions.*

(i) $K_r(a, b) \leq (\eta + \varepsilon)\, r$.

(ii) *For every $(u, v) \in B_{2^{-m}}(a, b)$ such that $ux + v = ax + b$,*

$$K_r(u, v) \geq (\eta - \varepsilon)\, r + \delta \cdot (r - t)\,,$$

whenever $t = -\log \|(a, b) - (u, v)\| \in (0, r]$.

[1] Although effective Hausdorff was originally defined by J. Lutz [6] using martingales, it was later shown by Mayordomo [12] that the definition used here is equivalent. For more details on the history of connections between Hausdorff dimension and Kolmogorov complexity, see [3, 13].

Then for every oracle set $A \subseteq \mathbb{N}$,

$$K_r^A(a, b, x \mid x, ax + b) \leq K_{m,r}^A(a, b \mid x, ax + b) + \frac{4\varepsilon}{\delta} r - K(\varepsilon, \eta) + O_{a,b,x}(\log r)\,.$$

Note that, by the symmetry of information, the conclusion of the above lemma can be equivalently written as

$$K_r^A(x, ax + b) \geq K_r^A(a, b, x) - K_{m,r}^A(a, b \mid x, ax + b) + \tfrac{4\varepsilon}{\delta} r - O_{a,b,x}(\log r)$$

The second lemma which will be important in proving our main theorem is the following.

Lemma 3 ([11])**.** *Let $a, b, x \in \mathbb{R}$. For all $u, v \in B_1(a, b)$ such that $ux + v = ax + b$, and for all $r \geq t := -\log \|(a, b) - (u, v)\|$,*

$$K_r(u, v) \geq K_t(a, b) + K_{r-t,r}(x|a, b) - O_{a,b,x}(\log r)\,.$$

This lemma shows that we can give a lower bound on the complexity of any line intersecting $(x, ax + b)$.

Finally, we also need the following oracle construction of Lutz and Stull.

Lemma 4 ([11])**.** *Let $r \in \mathbb{N}$, $z \in \mathbb{R}^2$, and $\eta \in \mathbb{Q} \cap [0, \dim(z)]$. Then there is an oracle $D = D(r, z, \eta)$ satisfying*

(i) *For every $t \leq r$, $K_t^D(z) = \min\{\eta r, K_t(z)\} + O(\log r)$.*

(ii) *For every $m, t \in \mathbb{N}$ and $y \in \mathbb{R}^m$, $K_{t,r}^D(y|z) = K_{t,r}(y|z) + O(\log r)$ and $K_t^{z,D}(y) = K_t^z(y) + O(\log r)$.*

3 Proof of the Dimension Spectrum Conjecture

3.1 Low Dimensional Lines

In this section, we prove the spectrum conjecture for lines with $\dim(a, b) \leq 1$.

Theorem 5. *Let $(a, b) \in \mathbb{R}^2$ be a slope-intercept pair with $\dim(a, b) \leq 1$.. Then for every $s \in [0, 1]$, there is a point $x \in \mathbb{R}$ such that*

$$\dim(x, ax + b) = s + \dim(a, b).$$

We will use many of the results of this section for the case when $\dim(a, b) > 1$, so we will state our claims in full generality. Fix a slope-intercept pair (a, b), and a real $s \in (0, 1)$. Let $d = \dim(a, b)$. Let $y \in \mathbb{R}$ be random relative to (a, b). Thus, for every $r \in \mathbb{N}$,

$$K_r^{a,b}(y) \geq r - O(\log r).$$

Define the sequence of natural numbers $\{h_j\}_{j\in\mathbb{N}}$ inductively as follows. Define $h_0 = 1$. For every $j > 0$, let

$$h_j = \min\left\{h \geq 2^{h_{j-1}} : K_h(a,b) \leq \left(d + \frac{1}{j}\right) h\right\}.$$

Note that h_j always exists. For every $r \in \mathbb{N}$, let

$$x[r] = \begin{cases} a[r - \lfloor sh_j \rfloor] & \text{if } r \in (\lfloor sh_j \rfloor, h_j] \text{ for some } j \in \mathbb{N} \\ y[r] & \text{otherwise} \end{cases}$$

where $x[r]$ is the rth bit of x. Define $x \in \mathbb{R}$ to be the real number with this binary expansion.

One of the most important aspects of our construction is that we encode (a subset of) the information of a into our point x. This is formalized in the following lemma.

Lemma 6. *For every $j \in \mathbb{N}$, and every r such that $sh_j < r \leq h_j$,*

$$K_{r-sh_j,r}(a, b \mid x, ax + b) \leq O(\log h_j).$$

Proof. By definition, the last $r - sh_j$ bits of x are equal to the first $r - sh_j$ bits of a. That is,

$$x[sh_j]\, x[sh_j + 1]\, \dots x[r] = a[0]\, a[1]\, \dots a[r - sh_j].$$

Therefore, since additional information cannot increase Kolmogorov complexity,

$$\begin{aligned} K_{r-sh_j,r}(a \mid x, ax + b) &\leq K_{r-sh_j,r}(a \mid x) \\ &\leq O(\log h_j). \end{aligned}$$

Note that, given $2^{-(r-sh_j)}$-approximations of a, x, and $ax + b$, it is possible to compute an approximation of b. That is,

$$K_{r-sh_j}(b \mid a, x, ax + b) \leq O(\log h_j).$$

Therefore, by Lemma 1,

$$\begin{aligned} K_{r-sh_j,r}(a, b \mid x, ax + b) &= K_{r-sh_j,r}(a \mid x, ax + b) \\ &\qquad + K_{r-sh_j,r}(b \mid a, x, ax + b) + O(\log r) \\ &\leq O(\log h_j) + K_{r-sh_j,r}(b \mid a, x, ax + b) + O(\log r) \\ &\leq O(\log h_j). \end{aligned}$$

□

The other important property of our construction is that (a, b) gives no information about x, beyond the information specifically encoded into x.

Lemma 7. *For every $j \in \mathbb{N}$, the following hold.*

1. $K_t^{a,b}(x) \geq t - O(\log h_j)$*, for all* $t \leq sh_j$*.*

2. $K_r^{a,b}(x) \geq sh_j + r - h_j - O(\log h_j)$*, for all* $h_j \leq r \leq sh_{j+1}$*.*

Proof. We first prove item (1). Let $t \leq sh_j$. Then, by our construction of x, and choice of y,

$$\begin{aligned} K_t^{a,b}(x) &\geq K_t^{a,b}(y) - h_{j-1} - O(\log t) \\ &\geq t - O(\log t) - \log h_j - O(\log t) \\ &\geq t - O(\log h_j). \end{aligned}$$

For item (2), let $h_j \leq r \leq sh_{j+1}$. Then, by item (1), Lemma 1 and our construction of x,

$$\begin{aligned} K_r^{a,b}(x) &= K_{h_j}^{a,b}(x) + K_{r,h_j}^{a,b}(x) - O(\log r) && \text{[Lemma 1]} \\ &\geq sh_j + K_{r,h_j}^{a,b}(x) - O(\log r) && \text{[Item (1)]} \\ &\geq sh_j + K_{r,h_j}^{a,b}(y) - O(\log r) \\ &\geq sh_j + r - h_j - O(\log r), \end{aligned}$$

and the proof is complete.

□

We now prove bounds on the complexity of our constructed point. We break the proof into two parts. In the first, we give lower bounds on $K_r(x, ax + b)$ at precisions $sh_j < r \leq h_j$.

Lemma 8. *For every* $\gamma > 0$ *and all sufficiently large* $j \in \mathbb{N}$*,*

$$K_r(x, ax + b) \geq (s + \min\{\dim(a, b), 1\})r - \gamma r,$$

for every $r \in (sh_j, h_j]$*.*

Proof. Let $d = \min\{\dim(a, b), 1\}$. Let $\eta \in \mathbb{Q}$ such that

$$d - \gamma/4 < \eta < d - \gamma^2.$$

Let $\varepsilon \in \mathbb{Q}$ such that

$$\varepsilon < \gamma(d - \eta)/16.$$

Let $D = D(r, (a, b), \eta)$ be the oracle of Lemma 4.

Let (u, v) be a line such that $t := \|(a, b) - (u, v)\| \geq r - sh_j$, and $ux + v = ax + b$. Note that $r - t \leq sh_j$. Then, by Lemma 3, Lemma 4 and Lemma 7(1),

$$\begin{aligned} K_r^D(u, v) &\geq K_t^D(a, b) + K_{r-t,r}^D(x \mid a, b) - O(\log r) && \text{[Lemma 3]} \\ &\geq K_t^D(a, b) + K_{r-t,r}(x \mid a, b) - O(\log r) && \text{[Lemma 4]} \\ &\geq dt + r - t - O(\log r) && \text{[Lemma 7(1)]} \\ &= \eta r + (1 - \eta)r - t(1 - d) - O(\log r) \\ &\geq \eta r + (1 - \eta)(r - t) - O(\log r) \\ &\geq (\eta - \varepsilon)r + (1 - \eta)(r - t). && (1) \end{aligned}$$

Therefore we may apply Lemma 2,

$$\begin{aligned} K_r(x, ax+b) &\geq K^D_r(a,b,x) - K^D_{r-sh_j,r}(a,b \,|, x, ax+b) && \text{[Lemma 2]} \\ &\quad - \frac{4\varepsilon}{1-\eta} r - O_{a,b,x}(\log r) \\ &\geq K^D_r(a,b,x) - K^D_{r-sh_j,r}(a,b \,|\, x, ax+b) - \frac{\gamma(d-\eta)}{4(1-\eta)} r - \frac{\gamma}{8} r \\ &= K^D_r(a,b,x) - K^D_{r-sh_j,r}(a,b \,|\, x, ax+b) - \frac{3\gamma}{8} r. && (2) \end{aligned}$$

By Lemma 7(1), our construction of oracle D, and the symmetry of information,

$$\begin{aligned} K^D_r(a,b,x) &= K^D_r(a,b) + K^D_r(x \,|\, a,b) - O(\log r) && \text{[Lemma 1]} \\ &= K^D_r(a,b) + K_r(x \,|\, a,b) - O(\log r) && \text{[Lemma 4(ii)]} \\ &\geq \eta r + K_r(x \,|\, a,b) - O(\log r) && \text{[Lemma 4(i)]} \\ &\geq \eta r + sh_j - \frac{\gamma}{4} r. && (3) \end{aligned}$$

Finally, by Lemma 6,

$$K^D_{r-sh_j,r}(a,b \,|, x, ax+b) \leq \frac{\gamma}{8} r. \qquad (4)$$

Together, inequalities (2), (3) and (4) imply that

$$\begin{aligned} K_r(x, ax+b) &\geq K^D_r(a,b,x) - K^D_{r-sh_j,r}(a,b \,|, x, ax+b) - \frac{3\gamma}{8} r \\ &\geq \eta r + sh_j - \frac{\gamma}{4} r - \frac{\gamma}{8} r - \frac{3\gamma}{8} r \\ &\geq dr - \frac{\gamma}{4} r + sh_j - \frac{3\gamma}{4} r \\ &\geq dr + sh_j - \gamma r \\ &\geq (s+d) r - \gamma r, \end{aligned}$$

and the proof is complete. □

We now give lower bounds on the complexity of our point, $K_r(x, ax+b)$, for $h_j < r \leq sh_{j+1}$.

Lemma 9. *For every $\gamma > 0$ and all sufficiently large $j \in \mathbb{N}$,*

$$K_r(x, ax+b) \geq (s + \min\{\dim(a,b), 1\}) r - \gamma r,$$

for every $r \in (h_j, sh_{j+1}]$.

Proof. Let $d = \min\{\dim(a,b), 1\}$, and $r \in (h_j, sh_{j+1}]$. We consider two cases, when $s \leq \dim(a,b)$ and when $s > \dim(a,b)$.

First assume that $s \leq \dim(a,b)$. Define

$$\alpha = \tfrac{s(r-h_j)+\dim(a,b)h_j}{r}.$$

Let $\eta \in \mathbb{Q} \cap (0, \alpha)$ such that

$$\alpha - \eta < (d-s)\gamma/4.$$

Let $\varepsilon \in \mathbb{Q}$ such that

$$\varepsilon < (\alpha - \eta)\gamma/16.$$

Finally, let $D = D(r, (a,b), \eta)$ be the oracle of Lemma 4.

Let (u,v) be a line such that $t := \|(a,b)-(u,v)\| \geq h_j$, and $ux+v = ax+b$. Then, by Lemmas 3, 4 and 7,

$$\begin{aligned}
K_r^D(u,v) &\geq K_t^D(a,b) + K_{r-t,r}^D(x \mid a,b) - O(\log r) && \text{[Lemma 3]}\\
&\geq K_t(a,b) + K_{r-t,r}(x \mid a,b) - O(\log r) && \text{[Lemma 4]}\\
&\geq \dim(a,b)t - o(t) + s(r-t) - O(\log r) && \text{[Lemma 7(1)]}\\
&= \dim(a,b)h_j + \dim(a,b)(t-h_j) + s(r-t) - o(r)\\
&= \dim(a,b)h_j + \dim(a,b)(t-h_j) + s(r-h_j) - s(t-h_j) - o(r)\\
&= \alpha r + (\dim(a,b)-s)(t-h_j) - o(r)\\
&= \eta r + (\alpha-\eta)r + (\dim(a,b)-s)(t-h_j) - o(r)\\
&\geq \eta r + (\alpha-\eta)(r-t) - o(r)\\
&\geq (\eta-\varepsilon)r + (\alpha-\eta)(r-t).
\end{aligned}$$

Therefore we may apply Lemma 2, which yields

$$\begin{aligned}
K_r^D(a,b,x \mid x, ax+b) &\leq K_{h_j,r}^D(a,b,x \mid x, ax+b) && \text{[Lemma 2]}\\
&\quad + \frac{4\varepsilon}{\alpha-\eta}r + K(\varepsilon,\eta) + O_{a,b,x}(\log r)\\
&\leq K_{h_j,r}^D(a,b,x \mid x, ax+b) + \frac{\gamma(\alpha-\eta)}{4(\alpha-\eta)}r + \frac{\gamma}{8}r\\
&= K_{h_j,r}^D(a,b,x \mid x, ax+b) + \frac{3\gamma}{8}r. && (5)
\end{aligned}$$

By Lemma 7, and our construction of oracle D,

$$\begin{aligned}
K_r^D(a,b,x) &= K_r^D(a,b) + K_r^D(x \mid a,b) - O(\log r) && \text{[Lemma 1]}\\
&= \eta r + K_r(x \mid a,b) - O(\log r) && \text{[Lemma 4]}\\
&\geq \eta r + sh_j + r - h_j - O(\log r) && \text{[Lemma 7(2)]}\\
&\geq \alpha r - \frac{\gamma}{4}r + sh_j + r - h_j - O(\log r)\\
&\geq s(r-h_j) + \dim(a,b)h_j - \frac{\gamma}{4}r + sh_j + r - h_j - O(\log r)\\
&\geq (1+s)r - (1-\dim(a,b))h_j - \frac{\gamma}{4}r. && (6)
\end{aligned}$$

By Lemmas 8, and 1, and the fact that additional information cannot increase Kolmogorov complexity

$$\begin{aligned}
K_{h_j,r}(a,b,x \mid x, ax+b) &\leq K_{h_j,h_j}(a,b,x \mid x, ax+b) \\
&= K_{h_j}(a,b,x) - K_{h_j}(x, ax+b) && \text{[Lemma 1]} \\
&\leq K_{h_j}(a,b) - dh_j + \frac{\gamma}{16}h_j && \text{[Lemma 8]} \\
&\leq \dim(a,b)h_j + h_j/j - dh_j + \frac{\gamma}{16}r \\
&\leq \dim(a,b)h_j - dh_j + \frac{\gamma}{8}r && (7)
\end{aligned}$$

Combining inequalities (5), (6) and (7) , we see that

$$\begin{aligned}
K^D_r(x, ax+b) &\geq K^D_r(a,b,x) - (\dim(a,b) - d)h_j - \frac{\gamma}{8}r - \frac{3\gamma}{8}r \\
&\geq (1+s)r - (1 - \dim(a,b))h_j - \frac{\gamma}{4}r \\
&\quad - (\dim(a,b) - d)h_j - \frac{3\gamma}{4}r \\
&= (1+s)r - h_j(1-d) - \gamma r.
\end{aligned}$$

Note that, since $d \leq 1$, and $h_j \leq r$,

$$\begin{aligned}
(1+s)r - h_j(1-d) - (s+d)r &= r(1-d) - h_j(1-d) \\
&= (r - h_j)(1-d) \\
&\geq 0.
\end{aligned}$$

Thus, since oracles cannot increase Kolmogorov complexity

$$\begin{aligned}
K_r(x, ax+b) &\geq K^D_r(x, ax+b) \\
&\geq (1+s)r - h_j(1-d) - \gamma r \\
&\geq (s+d)r - \gamma r,
\end{aligned}$$

and the proof is complete for the case $s \leq \dim(a,b)$.

The proof when $s > \dim(a,b)$ is essentially that of Lutz and Stull [11]. In this case, let $\eta \in \mathbb{Q} \cap (0, \dim(a,b))$ such that

$$\dim(a,b) - \eta < \tfrac{\gamma}{4}.$$

Let $\varepsilon \in \mathbb{Q}$ such that

$$\varepsilon < (s - \eta)\gamma/16.$$

Finally, let $D = D(r, (a,b), \eta)$ be the oracle of Lemma 4.

Let (u, v) be a line such that $t := \|(a, b) - (u, v)\| \geq 1$, and $ux + v = ax + b$. Then, by Lemmas 3, 4 and 7,

$$\begin{aligned}
K_r^D(u, v) &\geq K_t^D(a, b) + K_{r-t,r}^D(x \mid a, b) - O(\log r) && \text{[Lemma 3]} \\
&\geq \eta t + K_{r-t,r}(x \mid a, b) - O(\log r) && \text{[Lemma 4]} \\
&\geq \eta t + s(r - t) - O(\log r) && \text{[Lemma 7(1)]} \\
&= \eta t + \eta(r - t) + (s - \eta)(r - t) - O(\log r) \\
&= \eta r + (s - \eta)(r - t) + (1 - \eta)t - O(\log r) \\
&\geq (\eta - \varepsilon)r + (s - \eta)(r - t).
\end{aligned}$$

Therefore we may apply Lemma 2, which yields

$$\begin{aligned}
K_r^D(a, b, x \mid x, ax + b) &\leq K_{1,r}^D(a, b, x \mid x, ax + b) && \text{[Lemma 2]} \\
&\quad + \frac{4\varepsilon}{s - \eta} r + K(\varepsilon, \eta) + O_{a,b,x}(\log r) \\
&\leq \frac{\gamma(\alpha - \eta)}{4(s - \eta)} r + \frac{\gamma}{8} r \\
&= \frac{3\gamma}{8} r. && (8)
\end{aligned}$$

By Lemma 7, and our construction of oracle D,

$$\begin{aligned}
K_r^D(a, b, x) &= K_r^D(a, b) + K_r^D(x \mid a, b) - O(\log r) && \text{[Lemma 1]} \\
&= \eta r + K_r(x \mid a, b) - O(\log r) && \text{[Lemma 4]} \\
&\geq \eta r + s h_j + r - h_j - O(\log r) && \text{[Lemma 7(2)]} \\
&\geq dr - \frac{\gamma}{4} r + s h_j + r - h_j - O(\log r) \\
&\geq dr + s h_j + r - h_j - \frac{\gamma}{2} r. && (9)
\end{aligned}$$

Finally, by combining inequalities (8), (9) and (7) , we see that

$$\begin{aligned}
K_r^D(x, ax + b) &\geq K_r^D(a, b, x) - \frac{3\gamma}{8} r \\
&\geq dr + s h_j + r - h_j - \frac{\gamma}{2} r - \frac{3\gamma}{8} r \\
&\geq dr + sr - \gamma r,
\end{aligned}$$

and the proof is complete. □

We are now able to prove our main theorem.

Proof of Theorem 5. Let $(a, b) \in \mathbb{R}^2$ be a slope-intercept pair with $d = \dim(a, b) \leq 1$. Let $s \in [0, 1]$. Turetsky proved that $1 \in \mathrm{sp}(L_{a,b})$, hence, if $s = 0$, the conclusion holds.

If $s = 1$, then by [11], for any point x which is random relative to (a, b),

$$\dim(x, ax+b) = 1 + d,$$

and the claim follows.

Therefore, we may assume that $s \in (0,1)$. Let x be the point constructed in this section. Let $\gamma > 0$. Let j be large enough so that the conclusions of Lemmas 8 and 9 hold for these choices of (a,b), x, s and γ. Then, by Lemmas 8 and 9,

$$\begin{aligned}\dim(x, ax+b) &= \liminf_{r\to\infty} \frac{K_r(x, ax+b)}{r} \\ &\geq \liminf_{r\to\infty} \frac{(s+d)r - \gamma r}{r} \\ &= s + d - \gamma.\end{aligned}$$

Since we chose γ arbitrarily, we see that

$$\dim(x, ax+b) \geq s + d.$$

For the upper bound, let $j \in \mathbb{N}$ be sufficiently large. Then

$$\begin{aligned}K_{h_j}(x, ax+b) &\leq K_{h_j}(x, a, b) \\ &= K_{h_j}(a, b) + K_{h_j}(x \mid a, b) \\ &\leq dh_j + sh_j \\ &= (d+s)h_j.\end{aligned}$$

Therefore,

$$\dim(x, ax+b) \leq s + d,$$

and the proof is complete. □

3.2 High Dimensional Lines

Theorem 10. *Let $(a,b) \in \mathbb{R}^2$ be a slope-intercept pair with $\dim(a,b) > 1$.. Then for every $s \in [0,1]$, there is a point $x \in \mathbb{R}$ such that*

$$\dim(x, ax+b) = s + 1.$$

We will build on the construction, and results, of the previous section. We first slightly modify the construction. Let $y \in \mathbb{R}$ be random relative to (a,b). Define the sequence of natural numbers $\{h_j\}_{j\in\mathbb{N}}$ inductively as follows. Define $h_0 = 1$. For every $j > 0$, let

$$h_j = \min\left\{h \geq 2^{h_{j-1}} : K_h(a,b) \leq \left(d + \frac{1}{j}\right) h\right\}.$$

Note that h_j always exists. For every $r \in \mathbb{N}$, let

$$x[r] = \begin{cases} 0 & \text{if } r \in (\lfloor sh_j \rfloor, h_j] \text{ for some } j \in \mathbb{N} \\ y[r] & \text{otherwise} \end{cases}$$

where $x[r]$ is the rth bit of x. Define $x \in \mathbb{R}$ to be the real number with this binary expansion.

Fix a $j \in \mathbb{N}$ which is sufficiently large. Let $m = h_j - sh_j + 1$. We now define the set of points $x_0, \ldots, x_m$ by

$$x_n = x + 2^{-h_j+n}/a.$$

A useful fact about each point x_n is that

$$ax_n + b = ax + b + 2^{-h_j+n} \tag{10}$$

Lemma 11. *For every n, r such that $0 \leq n \leq m$ and $sh_j \leq r \leq h_j$ the following hold.*

1. $K_{n,h_j}(a \mid x_n) \leq \log(n)$.
2. *For every $n' > n$,*

$$|K_r(x'_n, ax'_n + b) - K_r(x_n, ax_n + b)| < n' - n + \log(r) + \log(r).$$

Proof. By our construction of x and x_n, the bits

$$x_n[h_j - n] \ldots x_n[h_j]$$

are the first n bits of $1/a$. Since we can compute a 2^{-n}-approximation of a given a 2^{-n}-approximation of $1/a$, the first claim follows.

For item (2), first assume that $r < h_j - n'$. Then

$$K_r(x_{n'}, ax_{n'} + b) = K_r(x_n, ax_n + b),$$

and the claim follows. Now assume that $r \geq h_j - n'$. By property (10),

$$K_r(x_{n'}, ax_{n'} + b) = K_r(\mathbf{a}, x_n, ax_n + b) + O(\log n),$$

where $\mathbf{a}$ is binary string of length $n' - n$ encoding the bits

$$a[r - (h_j - n)] \ldots a[r - (h_j - n')].$$

Therefore,

$$\begin{aligned} K_r(x_{n'}, ax_{n'} + b) &= K_r(\mathbf{a}, x_n, ax_n + b) + O(\log n) \\ &= K_r(x_n, ax_n + b) + K_r(\mathbf{a} \mid x_n, ax_n + b) + O(\log n). \end{aligned}$$

Since

$$K_r(\mathbf{a} \mid x_n, ax_n + b) \leq n' - n,$$

the proof of the second item is complete. □

We will use (a discrete, approximate, version of) the mean value theorem. For each n, define

$$M_n = \min\{\tfrac{K_r(x_n, ax_n+b)}{r} \mid sh_j \leq r \leq h_j\}.$$

The proof of our theorem essentially reduces to showing that, for some n,

$$1 + s - \epsilon \leq M_n \leq 1 + s + \epsilon. \tag{11}$$

We first note, by Lemma 8, that

$$K_r(x_m, ax_m + b) \gtrsim (1 + s)r - \gamma r,$$

for every $sh_j \leq r \leq h_j$, and so $M_m \gtrsim (1 + s)$. We also have the fact that

$$K_{h_j}(x, ax + b) \leq (1 + s)h_j.$$

That is, we have

$$M_0 \leq 1 + s \leq M_m.$$

If either inequality is not strict then we have shown that (11) holds and we are done. We may therefore assume that

$$M_0 < 1 + s - \epsilon, \text{ and } 1 + s + \epsilon < M_m.$$

Define

$$\begin{aligned} L &= \{n \mid M_n < 1 + s - \epsilon\} \\ G &= \{n \mid M_n > 1 + s + \epsilon\}. \end{aligned}$$

By our assumption, L and G are non-empty. Suppose that L and G partition $\{0, \ldots, m\}$. Then there is a n such that (without loss of generality) $n \in L$ and $n + 1 \in G$. However, by Lemma 11,

$$|K_r(x_{n+1}, ax_{n+1} + b) - K_r(x_n, ax_n + b)| < 1 + O(\log(r)),$$

for every r. Let r be a precision testifying to $x_n \in L$. Then

$$\begin{aligned} 1 + O(\log(r)) &> |K_r(x_n, ax_n + b) - K_r(x_n, ax_n + b)| \\ &> K_r(x_{n'}, ax_{n'} + b) - (1 + s - \epsilon)r \\ &> (1 + s + \epsilon)r - (1 + s - \epsilon)r \\ &= 2\epsilon r, \end{aligned}$$

which is false for all sufficiently large r. Therefore, we can conclude that (11) holds for some n.

We now fix an n such that

$$K_r(x_n, ax_n + b) \geq (1 + s)r - \epsilon r \tag{12}$$

for every $sh_j \leq r \leq h_j$.

We will modify x_n on the bits larger than h_j as follows. For every $h_j \leq r \leq sh_{j+1}$, let

$$x_n[r] = y[r].$$

It is important to note that changing bits larger than h_j does not change the complexity of x_n, nor $ax_n + b$, at precisions less than h_j. To avoid notational inconvenience, we will now refer to this point as x.

To complete the proof, let $r \geq h_j$. Then, by essentially the argument of Lemma 9, we can prove the following.

Lemma 12. *For every $\gamma > 0$ and all sufficiently large $j \in \mathbb{N}$,*

$$K_r(x, ax + b) \geq (s + 1)r - \gamma r,$$

for every $r \in (h_j, sh_{j+1}]$.

Proof. Define

$$\alpha = \tfrac{s(r-h_j)+\dim(a,b)h_j}{r}.$$

Let $\eta \in \mathbb{Q} \cap (0, \alpha)$ such that

$$\alpha - \eta < (1 - s)\gamma/4.$$

Let $\varepsilon \in \mathbb{Q}$ such that

$$\varepsilon < (\alpha - \eta)\gamma/16.$$

Finally, let $D = D(r, (a, b), \eta)$ be the oracle of Lemma 4.

Let (u, v) be a line such that $t := \|(a, b) - (u, v)\| \geq h_j$, and $ux + v = ax + b$. Then, by Lemmas 3, 4 and 7,

$$\begin{aligned}
K^D_r(u, v) &\geq K^D_t(a, b) + K^D_{r-t,r}(x \mid a, b) - O(\log r) && \text{[Lemma 3]} \\
&\geq K_t(a, b) + K_{r-t,r}(x \mid a, b) - O(\log r) && \text{[Lemma 4]} \\
&\geq \dim(a, b)t - o(t) + s(r - t) - O(\log r) && \text{[Lemma 7(1)]} \\
&= \dim(a, b)h_j + \dim(a, b)(t - h_j) + s(r - t) - o(r) \\
&= \dim(a, b)h_j + \dim(a, b)(t - h_j) + s(r - h_j) - s(t - h_j) - o(r) \\
&= \alpha r + (\dim(a, b) - s)(t - h_j) - o(r) \\
&= \eta r + (\alpha - \eta)r + (\dim(a, b) - s)(t - h_j) - o(r) \\
&\geq \eta r + (\alpha - \eta)(r - t) - o(r) \\
&\geq (\eta - \varepsilon)r + (\alpha - \eta)(r - t).
\end{aligned}$$

Therefore we may apply Lemma 2, which yields

$$\begin{aligned}
K^D_r(a, b, x \mid x, ax + b) &\leq K^D_{h_j,r}(a, b, x \mid x, ax + b) && \text{[Lemma 2]} \\
&\quad + \frac{4\varepsilon}{\alpha - \eta} r + K(\varepsilon, \eta) + O_{a,b,x}(\log r) \\
&\leq K^D_{h_j,r}(a, b, x \mid x, ax + b) + \frac{\gamma(\alpha - \eta)}{4(\alpha - \eta)} r + \frac{\gamma}{8} r \\
&= K^D_{h_j,r}(a, b, x \mid x, ax + b) + \frac{3\gamma}{8} r. && (13)
\end{aligned}$$

By Lemma 7, and our construction of oracle D,

$$\begin{aligned}
K^D_r(a,b,x) &= K^D_r(a,b) + K^D_r(x\,|\,a,b) - O(\log r) && \text{[Lemma 1]}\\
&= \eta r + K_r(x\,|\,a,b) - O(\log r) && \text{[Lemma 4]}\\
&\geq \eta r + sh_j + r - h_j - O(\log r) && \text{[Lemma 7(2)]}\\
&\geq \alpha r - \frac{\gamma}{4}r + sh_j + r - h_j - O(\log r)\\
&\geq s(r-h_j) + \dim(a,b)h_j - \frac{\gamma}{4}r + sh_j + r - h_j - O(\log r)\\
&\geq (1+s)r - (1-\dim(a,b))h_j - \frac{\gamma}{4}r && (14)\\
&\geq (1+s)r - \frac{\gamma}{4}r. && (15)
\end{aligned}$$

By Lemmas 1, inequality (12), and the fact that additional information cannot increase Kolmogorov complexity

$$\begin{aligned}
K_{h_j,r}(a,b,x\,|\,x,ax+b) &\leq K_{h_j,h_j}(a,b,x\,|\,x,ax+b)\\
&= K_{h_j}(a,b,x) - K_{h_j}(x,ax+b) && \text{[Lemma 1]}\\
&\leq K_{h_j}(a,b,x) - (1+s)h_j + \frac{\gamma}{16}h_j && \text{[In. (12)]}\\
&\leq \dim(a,b)h_j + h_j/j + sh_j\\
&\qquad - (1+s)h_j + \frac{\gamma}{16}h_j\\
&\leq \dim(a,b)h_j + h_j/j - h_j + \frac{\gamma}{16}r\\
&\leq \dim(a,b)h_j - h_j + \frac{\gamma}{8}r && (16)
\end{aligned}$$

Combining inequalities (13), (15) and (16) , we see that

$$\begin{aligned}
K^D_r(x,ax+b) &\geq K^D_r(a,b,x) - (\dim(a,b)-1)h_j - \frac{\gamma}{2}r\\
&\geq (1+s)r - \frac{\gamma}{4}r - (\dim(a,b)-1)h_j - \frac{3\gamma}{4}r\\
&= (1+s)r - \gamma r.
\end{aligned}$$

Thus, since oracles cannot increase Kolmogorov complexity

$$\begin{aligned}
K_r(x,ax+b) &\geq K^D_r(x,ax+b)\\
&\geq (1+s)r - \gamma r,
\end{aligned}$$

and the proof is complete □

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