Research Statement

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My research is focused on the interaction between analysis and theoretical computer science. Typically, theoretical computer science is concerned with discrete objects, such as the natural numbers or finite graphs. However, there is a deep connection with classical analysis and the theory of computing.

1 Effective Dimension and Fractal Geometry

The greatest focus of my research, thus far, has been on algorithmic (effective) dimension, and its relation to questions in fractal geometry. Effective dimensions were originally defined by J. Lutz [9, 10] and were subsequently characterized using Kolmogorov complexity by Mayordomo [16] and Athreya, Hitchcock, Lutz and Mayordomo [1]. The two principal notions of effective dimension are the effective (Hausdorff) dimension, $\dim(x)$, and effective packing dimension, Dim. These quantities can be thought of as the *density of algorithmic information* of a point in Euclidean space. Effective dimension is a widely studied concept, and has proved to be geometrically meaningful. Apart from being intrinsically interesting, from the point of view of theoretical computer science and information theory, effective dimension is made even more fascinating due to its strong connections to important problems in the field of fractal geometry. Indeed, one of my most important results, joint with Neil Lutz, uses algorithmic dimension to improve the best known bounds of Furstenberg sets, an important problem in fractal geometry.

The field of fractal geometry studies the fine-grained structure of irregular sets using different notions of (fractal) dimensions. Of particular importance are the Hausdorff dimension, $\dim_H(E)$, and the packing dimension, $\dim_P(E)$, of sets $E \subseteq \mathbb{R}^n$. Intuitively, these dimensions are alternative notions of size that allow us to quantitatively classify sets of measure zero. It is now a growing field within mathematical analysis, with several important and far reaching open problems.

A recent result (the point-to-set principle) of J. Lutz and N. Lutz [11] shows that the *classical* notions of Hausdorff and packing dimension can alternatively be defined using the effective notions of dimension. This result states that the Hausdorff (resp. packing) dimension of a *set* is the minimum over all oracles of the supremum of the effective (resp. strong) dimension of the *points* in the set. In other words, in order to understand the (classical) dimension of a set, we can try to understand the (effective) dimension of the points in a set.

1.1 Dimension of Points on Lines

The single largest portion of my research has been on the algorithmic dimension of points on lines. I will first highlight my work in this direction from a purely information theoretic point of view, and, afterwards, describe the connection to fractal geometry.

Given the point-wise nature of effective dimension, it is natural to investigate the *dimension spectrum* of a set $E \subseteq \mathbb{R}^n$; i.e., the set $\text{sp}(E) := \{\dim(x) : x \in E\}$. Even for as simple a set as a line, the structure of the dimension spectrum is not at all obvious (in fact, as discussed below, this is a very deep problem). My first result in this area, joint with Neil Lutz, gives the best possible lower bounds for "typical" points on any given line $L_{a,b}$ with slope a and intercept b. As a result of this, we were able to answer a longstanding open question originally posed by J. Lutz.

Question 0.1. Is there a straight line $L \subseteq \mathbb{R}^2$ such that every point on L has effective Hausdorff dimension 1; i.e., is there a line $L_{a,b}$ such that $\text{sp}(L_{a,b}) = 1$?

An immediate corollary of our theorem shows that this is impossible.

In subsequent work on the dimensions of points on lines, I focused on the following conjecture of J. Lutz.

Conjecture 0.1. Does every planar line $L_{a,b}$ have a dimension spectrum containing an interval of length 1?

This question, although easy to state, seems quite difficult. This is due to its intimate connection with an important open problem in fractal geometry (see below). However, we now know that Lutz's conjecture does hold for a wide variety of lines. In a paper with N. Lutz [13], we showed that Lutz's conjecture is true, assuming that the effective and strong dimension of the slope-intercept pair coincide; i.e., $\dim(a, b) = \text{Dim}(a, b)$. In a recent paper, I was able to show that Lutz's conjecture is also true for all lines whose slope-intercept pair has dimension of at least 1 ($\dim(a, b) \geq 1$) [20].

In a related direction, although not directly involving Lutz's conjecture, I have also investigated the question of proving *lower bounds* on the dimension spectrum of planar lines. In this direction, very little is known, for the same reason that it is intimately connected to the Furstenberg set conjecture. However, I was able to show [20] that for every line $L_{a,b}$ with slope a and intercept b, "most" points have complexity of at least $\frac{\dim(a,b)}{2}$. This is the first step of a very exciting research project, giving tight lower bounds of $\mathrm{sp}(L_{a,b})$.

1.1.1 Relation to Fractal Geometry

Many longstanding conjectures of geometric measure theory concern the classical dimensions (both Hausdorff and packing) of sets involving lines. Among the most prominent problems are the Kakeya and Nikodym conjectures, which are still open for dimensions $n > 2$, as well as the Furstenberg set conjecture. As discussed previously, the point-to-set principle gives tools which allow theoretical computer scientists to contribute to these investigations. Recent work has shown the viability of this program. In particular, in our paper [12], we improve the bounds of Furstenberg sets due to Molter and Rela [19] for certain classes of sets. Furthermore, we are also able to give new proofs of the Kakeya and Nikodym problems for $\mathbb{R}^2$. My recent work [20] on lower bounds of the dimension spectrum of lines can also be combined with the point-to-set principle to give a new, entirely information-theoretic, proof of Molter and Rela's results.

1.2 Dimension of Projected Points

A fundamental problem in fractal geometry is determining how (orthogonal) projection mappings affect dimension. Recently, Neil Lutz and I used algorithmic information theory, via the point-to-set principle, to study the Hausdorff and packing dimensions of orthogonal projections onto lines [14]. The starting point of our investigation was the celebrated Marstrand projection theorem [15]. This result states that, if $E \subseteq \mathbb{R}^2$ is analytic, then for *almost all* $e \in S^1$, the Hausdorff dimension of $\mathrm{proj}_e E$ is maximal.

It is natural to ask whether the requirement that E be analytic can be removed. Without further conditions, it cannot. Davies [5] showed that, assuming the continuum hypothesis, there are non-analytic sets for which Marstrand's theorem fails. However, we showed that if the Hausdorff and packing dimensions of E agree, then we can remove the requirement that E be analytic. We were also able to give nontrivial lower bounds on the *packing* dimension of the projections of *arbitrary* (i.e., not necessarily analytic) sets. This is, to the best of my knowledge, the best bound of the packing dimension for arbitrary sets. Finally, we gave a new, information-theoretic, proof of Marstrand's projection theorem. I believe that aspects of this proof are interesting in their own right. It is possible that the theoretical computer science viewpoint can shed light on the exact nature of the analytic requirement (as shown to be necessary by Davies).

2 Computable Analysis

2.1 Categoricity

I am interested in a classical area of computability theory, computable categoricity, in the context of computable analysis. Computable categoricity studies the computability of isomorphisms between discrete structures, and has historically been studied for discrete structures such as $\mathbb{N}$ and $\mathbb{Q}$. Recent work has studied categoricity in the context of computable analysis. For example, Tim McNicholl showed that, for every computable real $p \neq 2$, ℓ^p is *not* computably categorical [17].

In joint work with Tim McNicholl, we studied the isometry degree of ℓ^p [18]. The isometry degree is the least powerful oracle that computes an isomorphism between any two of its computable copies. We show that, when p is a computable real so that $p \geq 1$ and $p \neq 2$, the isometry degrees of the computable copies of ℓ^p are precisely the c.e. degrees.

The extension of computable categoricity to continuous domains opens many new avenues of research. Of immediate interest is the question of which structures of computable analysis are computably categorical. In joint work with Tim McNicholl and Joe Clanin [4], we investigated this question for L^p spaces. In contrast to the situation of ℓ^p, we show that L^p *is* computably categorical.

As a theoretical computer scientist, I am also interested in these questions in the context of resource-bounded computable analysis. I intend to study polynomial time and space categoricity for a variety of continuous structures.

2.2 Resource-bounded Randomness and Analysis

A recent line of investigation has shown a deep connection between algorithmic randomness and measure-theoretic analysis, mediated by computable analysis. In measure-theoretic analysis, theorems state that a property P holds for *almost every* real number; i.e., P holds except for an exceptional set of measure zero. By adding computability restrictions, researchers have been able to characterize various notions of algorithmic randomness using classical theorems from measure-theoretic analysis. As a theoretical computer scientist, I am interested in whether this connection holds in the resource-bounded setting. That is, can we characterize polynomial time or space randomness using measure theoretic analysis? While there have been some results showing this connection does exist, the question is still not as well understood as in the computable setting. In joint work with Xiang Huang, we addressed this question in the context of polynomial space randomness [7]. We first defined a new notion of polynomial space randomness, weak PSPACE randomness. We showed that Lutz PSPACE randomness implies weak PSPACE randomness. We then studied the connection between weak PSPACE randomness and the Lebesgue Differentiation Theorem. In the computable setting, the Lebesgue Differentiation Theorem has been used to characterize Schnorr randomness. We proved that this connection persists in the polynomial space setting by showing that the Lebesgue Differentiation Theorem characterizes weak PSPACE randomness.

The new notion of polynomial space randomness, weak PSPACE randomness, that Xiang Huang and I have defined leaves many directions for future research. One of the most promising is its connection to complexity theory. Lutz originally defined his notion of resource-bounded randomness in order to study complexity theory [8]. This direction has proven to be very fruitful, and is the basis of the "Lutz hypothesis" in complexity theory. I would like to pursue an analogous direction in the context of weak PSPACE randomness. A closely related direction is to extend weak PSPACE randomness to other computational resources. For example, I would like to define and investigate weak polynomial *time* randomness.

2.3 Semicomputable Geometry

An elementary, but difficult, question in computable analysis is the characterization of compact sets of various computational hardnesses. In recent joint work with Mathieu Hoyrup and Diego Saucedo [6], we studied this question in the context of *semicomputable* filled triangles in the plane. Even in this very restricted setting, the problem is surprisingly deep. In order to study this question, we introduced a notion of semicomputable points, a generalization of left c.e. real numbers to two dimensions. By studying the properties of semicomputable *points*, we were able to give a characterization of semicomputable *triangles* based on their vertex points. In doing so, we connected the definition of semicomputable points to the idea of Solovay reducibility from computable analysis. In other directions, we used computable genericity to give lower bounds of the computational difficulty for a wide range of parameters of semicomputable triangles.

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