# Results on the Dimension Spectra of Planar Lines

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**Abstract**

In this paper we investigate the (effective) dimension spectra of lines in the Euclidean plane. The dimension spectrum of a line $L_{a,b}$, $\text{sp}(L)$, with slope $a$ and intercept $b$ is the set of all effective dimensions of the points $(x, ax + b)$ on $L$. It has been recently shown that, for every $a$ and $b$ with effective dimension less than 1, the dimension spectrum of $L_{a,b}$ contains an interval. Our first main theorem shows that this holds for every line. Moreover, when the effective dimension of $a$ and $b$ is at least 1, $\text{sp}(L)$ contains a *unit* interval.

Our second main theorem gives lower bounds on the dimension spectra of lines. In particular, we show that for every $\alpha \in [0, 1]$, with the exception of a set of Hausdorff dimension at most $\alpha$, the effective dimension of $(x, ax+b)$ is at least $\alpha + \frac{\dim(a,b)}{2}$. As a consequence of this theorem, using a recent characterization of Hausdorff dimension using effective dimension, we give a new proof of a result by Molter and Rela on the Hausdorff dimension of Furstenberg sets.

## 1 Introduction

This paper is concerned with the algorithmic dimension of points on a given line in the Euclidean plane. The most well-studied algorithmic dimensions for a point $x \in \mathbb{R}^n$ are the *effective Hausdorff dimension*, $\dim(x)$, and its dual, the *effective packing dimension*, $\text{Dim}(x)$ [7, 1]. Given the pointwise nature of effective dimension, it is natural to consider the dimension spectrum, $\text{sp}(A)$, of a set $A \subseteq \mathbb{R}^n$, which is defined to be the set of $\dim(x)$ for all $x \in A$.

In this paper, we study the behavior of $\text{sp}(L_{a,b})$, where $L_{a,b}$ is the line with slope $a$ and intercept $b$. Turetsky [19] gave the first result on the dimension spectra of lines, showing that, for every $n \geq 2$, the set of all points in $\mathbb{R}^n$ with effective Hausdorff 1 is connected, guaranteeing that $1 \in \text{sp}(L_{a,b})$. It was then asked by J. Lutz, with the expectation of a negative answer, if there were lines in the plane whose dimension spectrum was the singleton $\{1\}$. N. Lutz and Stull [13] showed that this cannot happen by proving the following theorem.

| | |
|---|---|
| $\forall a, b$ | $1 \in \mathrm{sp}(L_{a,b})$ [19] |
| $\dim(a, b) = 2$ | $\mathrm{sp}(L_{a,b}) = [1, 2]$ |
| $\dim(a, b) \geq 1$ | $\mathrm{sp}(L_{a,b})$ infinite [14] |
| $\dim(a, b) = d < 1$ | $[2d, 1 + d] \subseteq \mathrm{sp}(L_{a,b})$ [13] |
| $\dim(a, b) = 0$ | $\mathrm{sp}(L_{a,b}) = [0, 1]$ |
| $\dim(a, b) = \mathrm{Dim}(a, b) = d$ | $[\min\{1, d\}, 1 + \min\{1, d\}] \subseteq \mathrm{sp}(L_{a,b})$ [14] |

Table 1: Previously known results about the dimension spectra of lines.

**Theorem 1** (N. Lutz and Stull [13]). *For all $a, b, x \in \mathbb{R}$,*

$$\dim(x, ax + b) \geq \dim^{a,b}(x) + \min\{\dim(a, b), \dim^{a,b}(x)\} .$$

Together with the fact that $\dim(a, b) = \dim(a, a^2 + b) \in \mathrm{sp}(L_{a,b})$ and Turetsky's result, this implies that the dimension spectrum of $L_{a,b}$ contains both endpoints of the unit interval $[\min\{1, \dim(a, b)\}, \min\{1, \dim(a, b)\} + 1]$. Theorem 1 also implies that, when $\dim(a, b) < 1$, the dimension spectrum of $L_{a,b}$ contains the interval $[2\dim(a, b), \dim(a, b) + 1]$. With this result, it is natural to conjecture that the dimension spectrum of every line $L_{a,b}$ contains an interval. Indeed, in a recent survey on effective dimension, N. Lutz [11] proposed the question of whether *every* line $L_{a,b}$ has a dimension spectrum containing a *unit* interval. Building upon the techniques of [13], N. Lutz and Stull [14] showed that this is the case for a restricted class of lines.

**Theorem 2** (N. Lutz and Stull [14]).

1. *If $\dim(a, b) = \mathrm{Dim}(a, b)$, then $\mathrm{sp}(L_{a,b})$ contains a unit interval.*
2. *If $\dim(a, b) \geq 1$, then $\mathrm{sp}(L_{a,b})$ is infinite.*

The second item, combined with Theorem 1, shows that for every line $L_{a,b}$, the dimension spectrum of $L_{a,b}$ is infinite. Table 1 gives a summary of these results.

The question of whether the dimension spectrum of every line contains an interval has remained open. Our first main theorem settles this question.

**Theorem 3.** *Let $(a, b) \in \mathbb{R}^2$ such that $\dim(a, b) \geq 1$. Then, for every real number $d \in [0, 1]$, there is a point $x$ such that*

$$\dim(x, ax + b) = 1 + d.$$

Our second main theorem deals with providing lower bounds on the dimension spectrum of a given line in the plane. The previously discussed theorems have all focused on results proving that the spectrum of a given line contains certain values. However, very little is known about the lower bound of $\text{sp}(L_{a,b})$ for arbitrary lines $L_{a,b}$. Our second main theorem gives a lower bound of the spectrum of arbitrary lines, disregarding a set of small Hausdorff dimension.

**Theorem 4.** *For every $a, b \in \mathbb{R}$ and $\alpha \in (0,1)$, the set*

$$A = \{x \mid dim(x, ax+b) \leq \alpha + \tfrac{\dim(a,b)}{2}\}$$

*has Hausdorff dimension at most $\alpha$.*

Apart from being intrinsically interesting, the study of the effective dimension of points on a line has strong connections to important problems in the field of Fractal Geometry. This connection is mediated by the following theorem relating the two notions of the effective dimension of points with the Hausdorff and packing dimension of sets.

**Theorem 5** (Point-to-set principle [8]). *Let $n \in \mathbb{N}$ and $E \subseteq \mathbb{R}^n$. Then*

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x), \text{ and}$$
$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \text{Dim}^A(x)\,.$$

Recent work has used effective dimension and the point-to-set principle to prove new results in Fractal Geometry [10, 12]. In particular, the point-to-set principle combined with Theorem 1 gives improved lower bounds on the Hausdorff dimension of a certain class of *Furstenberg sets* [13], an important open problem in Fractal Geometry (see Section 5 for definitions). As our final result, we show that our second main theorem, Theorem 4, gives a new proof of a result by Molter and Rela [17] on the dimension of Furstenberg sets.

# 2 Preliminaries

## 2.1 Kolmogorov Complexity in Discrete Domains

The *conditional Kolmogorov complexity* of $\sigma \in \{0,1\}^*$ given $\tau \in \{0,1\}^*$ is

$$K(\sigma|\tau) = \min_{\pi \in \{0,1\}^*} \{\ell(\pi) : U(\pi, \tau) = \sigma\}\,,$$

where $U$ is a fixed universal prefix-free Turing machine and $\ell(\pi)$ is the length of $\pi$. Any $\pi$ that achieves this minimum is said to *testify* to, or be a *witness* to, the value $K(\sigma|\tau)$. The *Kolmogorov complexity* of $\sigma$ is $K(\sigma) = K(\sigma|\lambda)$, where $\lambda$ is the empty string. An important property of Kolmogorov complexity is the *symmetry of information* (attributed to Levin in [4]):

$$K(\sigma|\tau, K(\tau)) + K(\tau) = K(\tau|\sigma, K(\sigma)) + K(\sigma) + O(1)\,.$$

These definitions and this symmetry extend naturally to other discrete domains (e.g., integers, rationals, etc.) via standard binary encodings. See [6, 18, 3] for detailed discussion of these topics.

## 2.2 Kolmogorov Complexity in Euclidean Spaces

The above definitions may also be lifted to Euclidean spaces by introducing variable precision parameters [9, 8]. Let $x \in \mathbb{R}^m$, and let $r, s \in \mathbb{N}$.[1] For $\varepsilon > 0$, $B_\varepsilon(x)$ denotes the open ball of radius $\varepsilon$ centered on $x$.

The *Kolmogorov complexity of $x$ at precision $r$* is

$$K_r(x) = \min\left\{K(p) \,:\, p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m\right\} .$$

The *conditional Kolmogorov complexity of $x$ at precision $r$ given $q \in \mathbb{Q}^m$* is

$$\hat{K}_r(x|q) = \min\left\{K(p) \,:\, p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m\right\} .$$

The *conditional Kolmogorov complexity of $x$ at precision $r$ given $y \in \mathbb{R}^n$ at precision $s$* is

$$K_{r,s}(x|y) = \max\left\{\hat{K}_r(x|q) \,:\, q \in B_{2^{-r}}(y) \cap \mathbb{Q}^n\right\} .$$

We abbreviate $K_{r,r}(x|y)$ by $K_r(x|y)$.

The following lemma shows that the above definitions are only linearly sensitive to their precision parameters. Intuitively, making an estimate of a point slightly more precise only requires a small amount of information.

**Lemma 6.** (Case and J. Lutz [2]) *There is a constant $c \in \mathbb{N}$ such that for all $n, r, s \in \mathbb{N}$ and $x \in \mathbb{R}^n$,*

$$K_r(x) \leq K_{r+s}(x) \leq K_r(x) + K(r) + ns + a_s + c\,,$$

*where* $a_s = K(s) + 2\log(\lceil \frac{1}{2}\log n\rceil + s + 3) + (\lceil \frac{1}{2}\log n\rceil + 3)n + K(n) + 2\log n$.

In Euclidean spaces, we have a weaker version of symmetry of information.

**Lemma 7** (J. Lutz and N. Lutz [8], N. Lutz and Stull [13])**.** *Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. For all $r, s \in \mathbb{N}$ with $r \geq s$,*

1. $K_r(x, y) = K_r(x|y) + K_r(y) + O(\log r)$.
2. $K_r(x) = K_{r,s}(x|x) + K_s(x) + O(\log r)$.

## 2.3 Effective Dimensions

Although effective Hausdorff dimension was initially developed by J. Lutz using generalized martingales [7], it was later shown by Mayordomo [15] that it may be equivalently defined as the lower asymptote of the density of algorithmic information. That is the characterization we use here. For more details on the history of connections between Hausdorff dimension and Kolmogorov complexity, see [3, 16].

[1] As a matter of notational convenience, if we are given a nonintegral positive real as a precision parameter, we will always round up to the next integer. For example, $K_r(x)$ denotes $K_{\lceil r\rceil}(x)$ whenever $r \in (0, \infty)$.

The *effective Hausdorff dimension* and *effective packing dimension* of a point $x \in \mathbb{R}^n$ are

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r} \quad \text{and} \quad \operatorname{Dim}(x) = \limsup_{r\to\infty} \frac{K_r(x)}{r}\,.$$

Intuitively, these dimensions measure the density of algorithmic information in the point $x$. Guided by the information-theoretic nature of these characterizations, J. Lutz and N. Lutz [8] defined the *lower* and *upper conditional dimension* of $x \in \mathbb{R}^m$ given $y \in \mathbb{R}^n$ as

$$\dim(x|y) = \liminf_{r\to\infty} \frac{K_r(x|y)}{r} \quad \text{and} \quad \operatorname{Dim}(x|y) = \limsup_{r\to\infty} \frac{K_r(x|y)}{r}\,.$$

### 2.4 Relative Complexity and Dimensions

By letting the underlying fixed prefix-free Turing machine $U$ be a universal *oracle* machine, we may *relativize* the definition in this section to an arbitrary oracle set $A \subseteq \mathbb{N}$. The definitions of $K^A(\sigma|\tau)$, $K^A(\sigma)$, $K^A_r(x)$, $K^A_r(x|y)$, $\dim^A(x)$, $\operatorname{Dim}^A(x)$ $\dim^A(x|y)$, and $\operatorname{Dim}^A(x|y)$ are then all identical to their unrelativized versions, except that $U$ is given oracle access to $A$.

We will frequently consider the complexity of a point $x \in \mathbb{R}^n$ *relative to a point* $y \in \mathbb{R}^m$, i.e., relative to a set $A_y$ that encodes the binary expansion of $y$ is a standard way. We then write $K^y_r(x)$ for $K^{A_y}_r(x)$. J. Lutz and N. Lutz showed that $K^y_r(x) \leq K_{r,t}(x|y) + K(t) + O(1)$ [8].

## 3 Dimension Spectra of Lines of High Dimension

### 3.1 Approach and Previous Work

In this section we state the technical lemmas that underlie the proof of our first main theorem (Theorem 3). These lemmas were first stated and proved by N. Lutz and Stull [13, 14][2].

**Lemma 8.** *Let $a, b, x \in \mathbb{R}$, $k \in \mathbb{N}$. Suppose that $r_1, \ldots, r_k \in \mathbb{N}$, $\delta \in \mathbb{R}_+$, and $\varepsilon, \eta_1, \ldots, \eta_k \in \mathbb{Q}_+$ satisfy the following conditions for every $1 \leq i \leq k$.*

1. $r_i \geq \log(2|a| + |x| + 6) + r_{i-1}$.
2. $K_{r_i}(a, b) \leq (\eta_i + \varepsilon)\, r_i$.
3. *For every $(u, v) \in \mathbb{R}^2$ such that $t = -\log \|(a, b) - (u, v)\| \in (r_{i-1}, r_i]$ and $ux + v = ax + b$, $K_{r_i}(u, v) \geq (\eta_i - \varepsilon)\, r_i + \delta \cdot (r_i - t)$.*

[2] Lemma 8 is stated here in a slightly stronger form than the version of [14]. The proof, however, is nearly identical. For completeness we give a proof in the Technical Appendix.

*Then for every oracle set $A \subseteq \mathbb{N}$,*

$$K^A_{r_k}(a,b,x \mid x, ax+b) \leq 2^k \left( K(\eta_1,\dots,\eta_k) + K(\varepsilon) + \frac{4\varepsilon}{\delta} r_k + O(\log r_k) \right) .$$

We will briefly describe the intuition behind Lemma 8. For $k = 1$, Lemma 8 roughly states that, if $x$ and $(a,b)$ satisfy the following properties, then we can compute an approximation of $(a,b)$ given an approximation of $(x, ax+b)$.

1. $K_r(a,b)$ is small.
2. For every $(u,v)$ such that $ux + v = ax + b$ either
   - $K_r(u,v)$ is large, or
   - $(u,v)$ is close to $(a,b)$

This follows outputting a $(u,v)$ of low complexity such that $ux+v = ax+b$. Under the above assumptions, any such pair must be close to $(a,b)$, and so we can recover $(a,b)$ with a small amount of extra information. Roughly, when $k > 1$, we do this procedure iteratively. That is, we begin by computing $(a,b)$ to precision $r_1$ in the manner described above. Having done so, we do the same procedure, except that we restrict to finding a pair $(u,v)$ *within* $2^{-r_1}$ of $(a,b)$, and so on. This is useful when we can only guarantee that $K_r(u,v)$ is large when $(u,v)$ is somewhat close to $(a,b)$, which is the case in the proof of Theorem 3.

The next two lemmas will ensure that item (1) and (2) hold for a given pair $(a,b)$.

**Lemma 9** (N. Lutz and Stull [13])**.** *Let $a,b,x \in \mathbb{R}$. For all $(u,v) \in \mathbb{R}^2$ such that $ux+v = ax+b$ and $t = -\log \|(a,b)-(u,v)\| \in (0,r]$,*

$$K_r(u,v) \geq K_t(a,b) + K^{a,b}_{r-t}(x) - O(\log r) .$$

**Lemma 10** (N. Lutz and Stull [14])**.** *Let $z \in \mathbb{R}^n$, $\eta \in \mathbb{Q} \cap [0, \dim(z)]$, and $k \in \mathbb{N}$. For all $r_1,\dots,r_k \in \mathbb{N}$, there is an oracle $D = D(r_1,\dots,r_k,z,\eta)$ such that*

1. *For every $t \leq r_1$, $K^D_t(z) = \min\{\eta r_1, K_t(z)\} + O(\log r_k)$*
2. *For every $1 \leq i \leq k$,*

$$K^D_{r_i}(z) = \eta r_1 + \sum_{j=2}^{i} \min\{\eta(r_j - r_{j-1}), K_{r_j,r_{j-1}}(z \mid z)\} + O(\log r_k) .$$

3. *For every $t \in \mathbb{N}$ and $x \in \mathbb{R}$, $K^{z,D}_t(x) = K^z_t(x) + O(\log r_k)$.*

### 3.2 First Main Theorem

In this section we prove our first main theorem, Theorem 3. To do so, we will break the proof into two cases. In the first we assume that, for arbitrarily long intervals, $K_r(a, b)$ is arbitrarily close to 1. In this case, it is "locally" as if $\dim(a, b) = \mathrm{Dim}(a, b)$, and we can use a similar proof to that of Theorem 2 in [14]. This case is formalized in the following lemma, whose proof if deferred to the appendix.

**Lemma 11.** *Let $(a, b) \in \mathbb{R}^2$ such that $\dim(a, b) \geq 1$. Assume that, for every $\tau > 0$ and every $M \in \mathbb{N}$ there are infinitely many $R \in \mathbb{N}$ such that*

$$K_s(a, b) \leq (1 + \tau)s,$$

*for every natural number $s \in [R, MR]$. Then, for every real number $d \in (0, 1]$, there is a point $x$ such that $\dim(x, ax + b) = 1 + d$.*

If $(a, b)$ is not of the first case, then there is a bound $(1 + \tau) > 1$ so that $K_r(a, b) \geq r(1 + \tau)$ for some $r$ in every sufficiently large interval. This implies that, for almost every precision $r$, the conditional complexity $K_{s,r}(a, b \,|\, a, b) > s - r$, for some $s$ at most a constant multiple of $r$. This fact allows us to use the procedure outlined in Section 3.1 at precision $s$. We will now formalize this intuition.

**Theorem 3.** *Let $(a, b) \in \mathbb{R}^2$ such that $\dim(a, b) \geq 1$. Then, for every real number $d \in [0, 1]$, there is a point $x$ such that*

$$\dim(x, ax + b) = 1 + d.$$

*Proof.* Let $(a, b) \in \mathbb{R}^2$ such that $\dim(a, b) \geq 1$. For $d = 1$, we may choose an $x \in \mathbb{R}$ that is random relative to $(a, b)$. That is, there is some constant $c \in \mathbb{N}$ such that for all $r \in \mathbb{N}$, $K^{a,b}_r(x) \geq r - c$. By Theorem 1,

$$\begin{aligned}\dim(x, ax + b) &\geq \dim^{a,b}(x) + \min\{\dim(a, b), 1\}\\ &= \liminf_{r\to\infty} \frac{K_r(x)}{r} + 1\\ &= 2,\end{aligned}$$

and the conclusion holds. For $d = 0$, the conclusion follows from Turetsky's theorem [19]. We therefore assume that $d \in (0, 1)$.

If $(a, b)$ satisfy the conditions of Lemma 11, then the conclusion is immediate. So assume that the conditions of Lemma 11 do not hold. Let $\tau > 0$ and $M > 0$ be constants such that, for almost every $R \in \mathbb{N}$,

$$K_s(a, b) > (1 + \tau)s,$$

for some $s \in [R, MR]$.

Let $y \in \mathbb{R}$ be random relative to $(a, b)$. Define the sequence of natural numbers $\{h_n\}_{n\in\mathbb{N}}$ inductively as follows. Define $h_0 = 1$. For every $n > 0$, let

$$h_n = \min\left\{h \geq 2^{h_{n-1}} : K_h(a, b) \geq \left(\mathrm{Dim}(a, b) - \frac{1}{n}\right) h\right\}.$$

Note that $h_n$ always exists. For every $r \in \mathbb{N}$, let

$$x[r] = \begin{cases} 0 & \text{if } \frac{r}{h_n} \in (d, 1] \text{ for some } n \in \mathbb{N} \\ y[r] & \text{otherwise} \end{cases}$$

where $x[r]$ is the $r$th bit of $x$. Define $x \in \mathbb{R}$ to be the real number with this binary expansion. Then $K_{dh_n}(x) = dh_n + O(\log dh_n)$.

*Claim 1:* $\dim(x, ax + b) \leq 1 + d$.

Let $\eta \in \mathbb{Q} \cap (0, 1)$, $\varepsilon \in \mathbb{Q}$, $n \in \mathbb{N}$, and let $m = \frac{1-d}{1-\eta}$. We first give lower bounds of the complexity of $K_r(x, ax+b)$ on the interval $(dh_n, mh_n)$. To begin, consider $r = h_n$. Let $k = \frac{r}{dh_n} = \frac{1}{d}$, and define $r_i = idh_n$ for every $1 \leq i \leq k$. As in the proof of Lemma 11, is important to note that $k$ is bounded by a constant depending only on $\eta$ and $d$.

*Claim 2:* $K_{h_n}(x, ax+b) \geq dh_n + \eta h_n - 2^k \left(K(\varepsilon) + kK(\eta) + \frac{4\varepsilon}{1-\eta} h_n + O(\log h_n)\right)$.

With this bound on the complexity of $(x, ax+b)$ at precision $h_n$, we will use a symmetry of information argument to give a lower bound on the complexity at precision $r \in (dh_n, h_n)$. We defer the proof of this claim to the appendix.

*Claim 3:* For all $r \in [dh_n, h_n)$,

$$K_r(x, ax + b) \geq r(d + \eta) - 2^k \left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1-\eta} h_n + O(\log h_n)\right).$$

Note that this lower bound is useful for $r \in (dh_n, h_n)$, since $h_n$ is a fixed constant multiple of $r$.

We now turn to proving lower bounds for the complexity of $(x, ax + b)$ on the interval $(h_n, mh_n)$. To do so, we will make use of our assumption that the complexity of $K_s(a, b)$ is at least $(1+\tau)s$. In particular, this assumption implies that there is a *fixed* constant $c$ such that $K_{ch_n, h_n}(a, b \mid a, b) \geq \eta(ch_n - h_n)$. Moreover, this constant is independent of $\eta$ and $\varepsilon$. To see this, let $s > h_n$ be a precision such that $K_s(a, b) \geq (1 + \tau)s$. By Lemma 7 and our assumption of $a, b$,

$$\begin{aligned} K_{s,h_n}(a, b \mid a, b) &\geq K_s(a, b) - K_{h_n}(a, b) - O(\log s) \\ &\geq (1 + \tau)s - \text{Dim}(a, b)h_n - O(\log s) \\ &\geq (1 + \tau)s - 2h_n - O(\log s). \end{aligned}$$

Thus,

$$K_{s,h_n}(a, b \mid a, b) \geq \eta(s - h_n),$$

for any such $s > ch_n$, for some fixed constant $c$ depending only on $\tau$. With this fact we are able to show the following, whose proof is deferred to the appendix.

*Claim 4:* There is a precision $h_n < j \leq ch_n$ such that

$$K_j(x, ax + b) \geq j - (h_n - dh_n) + \eta j - 2^k \left(K(\varepsilon) + kK(\eta) + \frac{4\varepsilon}{1-\eta} j + O(\log j)\right).$$

With this bound, we will again prove lower bounds at precisions $r \in (h_n, j)$ using symmetry of information arguments. While this is similar in spirit to the proof of Claim 3, there is an important difference. At precisions greater than $h_n$, the complexity of $x$ begins increasing again. In particular, the construction of $x$ and Lemma 6 implies the following.

$$K_j(x, ax+b) \leq K_r(x, ax+b) + 2(j-r).$$

We are still, however, able to achieve the required lower bounds on the complexity of $(x, ax+b)$ for all $r \in (h_n, j)$. The proof of this claim is deferred to the appendix.

*Claim 5:* For every $r \in (h_n, j)$,

$$K_r(x, ax+b) \geq r(d+\eta) - cr(1-\eta) - 2^k \left(K(\varepsilon) + K(\eta) + \tfrac{4\varepsilon}{1-\eta} cr + O(\log cr)\right).$$

This lower bound is useful since $j \leq ch_n$, and $c$ is a constant depending only on $\tau$. In particular, this allows us to make $\frac{cr(1-\eta)}{r}$ arbitrarily small by having $\eta$ go to 1.

To complete the proof for the interval $(dh_n, mh_n)$, we will apply the same method as in Claims 4 and 5, except that we use $j$ instead of $h_n$. Specifically, we choose the first $j_2 > j$ such that

$$K_{j_2,j}(a, b \mid a, b) \geq \eta(j_2 - j),$$

and note that $j_2 \leq cj$. We then apply the proof of Claim 4 to $K_{j_2}(x, ax+b)$, and the proof of Claim 5 to the interval $(j, j_2)$. We then repeat this argument until we have given the appropriate lower bound for all $r \in (h_n, mh_n)$.

Finally, taking Claims 1, 2, 3, 4 and 5 together yields the following. For every $r \in (dh_n, m_n)$,

$$K_r(x, ax+b) \geq r(d+\eta) - cr(1-\eta) - 2^k \left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1-\eta} cr + O(\log cr)\right). \tag{1}$$

To complete the proof, we bound $K_r(x, ax+b)$ for every $r \in [mh_n, dh_{n+1})$. By Lemma 7 and our construction of $x$,

$$\begin{aligned} K_r(x) &= K_{r,h_n}(x \mid x) + K_{h_n}(x) + o(r) \\ &= r - h_n + nh_n + o(r) \\ &\geq \eta r + o(r)\,. \end{aligned}$$

The proof of Theorem 1 gives $K_r(x, ax+b) \geq K_r(x) + \dim(x)r - o(r)$, and so $K_r(x, ax+b) \geq r(d+\eta) - \varepsilon r$. Combined with inequality (1), for every

$r \in (dh_n, dh_{n+1})$,

$$\begin{aligned}\frac{K_r(x, ax+b)}{r} &\geq r(d+\eta) - cr(1-\eta) - 2^k \left( K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1-\eta} cr + O(\log cr) \right) \\ &= d + \eta - 2^k \left( \frac{K(\varepsilon)}{r} + \frac{K(\eta)}{r} + \frac{4\varepsilon}{1-\eta} c + \frac{O(\log cr)}{r} \right) \\ &\geq d + \eta - 2^{\frac{m}{d}} \left( \frac{K(\varepsilon)}{r} + \frac{K(\eta)}{r} + \frac{4\varepsilon}{1-\eta} c + \frac{O(\log cr)}{r} \right)\end{aligned}$$

for all sufficiently large $n$. Since $\eta$ and $\varepsilon$ were chosen arbitrarily and independently,

$$\dim(x, ax+b) \geq d + 1,$$

and the proof is complete.

□

# 4 Lower Bounding the Dimension Spectrum of a Line

In this section we give the first nontrivial lower bounds of the dimension spectrum of an arbitrary line. For intuition behind the proof, first note the following simple observation.

**Observation 12.** *For every $x, y, a, b \in \mathbb{R}$,*

$$K_r(x, y, a, b) \leq K_r(x, ax+b) + K_r(y, ay+b) + 2t,$$

*where $t = -\log \|x - y\|$.*

Essentially, this is true since any two points identify a line, and this can be done in a computable way. The $2t$ extra information is due to the fact the precision which we can compute $(a, b)$ to is linearly correlated to the distance between $x$ and $y$. This immediately suggests an approach to give the lower bound

$$\dim(x) + \tfrac{\dim(a,b)}{2} \geq \dim(a, b).$$

While this observation is at the core of the proof of our second main theorem, it alone does not suffice. The principle issue is that the values of $K_r(x, ax+b)$, $K_r(y, ay+b)$ might be "out of phase"; that is, $K_r(x, ax+b)$ is small when $K_r(y, ay+b)$ is large, and vice versa. Our main theorem will show that the set of these points has low Hausdorff dimension.

Our first lemma builds upon Observation 12. In particular, it shows that, if $K_r(x, ax+b)$ is small, then every other $y$ such that $K_r(y, ay+b)$ is small must satisfy certain properties.

**Lemma 13.** *Let $\alpha \in (0,1)$, $x, a, b \in \mathbb{R}$, and $n, r \in \mathbb{N}$ such that $2r^{-\frac{1}{2}} < \frac{1}{n}$. Assume that $K_r(x, ax+b) < \alpha r + \frac{K_r(a,b)}{2} - \frac{r}{n}$, and $K_r^{a,b}(x) \geq \alpha r$. Then, for every $y \in \mathbb{R}$, if $K_r(y, ay+b) < \alpha r + \frac{K_r(a,b)}{2}$, at least one of the following holds.*

1. *$t := -\log \|x - y\| \leq r^{\frac{1}{2}}$.*
2. *$K_r(y \mid a, b, x) < \alpha r$.*

For every $a, b \in \mathbb{R}$ and $\alpha \in (0,1)$, define the set

$$A(\alpha, a, b) = \{x \mid \dim(x, ax+b) < \alpha + \tfrac{\dim(a,b)}{2}\}.$$

**Theorem 4.** *For every $a, b \in \mathbb{R}$ and $\alpha \in (0,1)$, $\dim_H(A(\alpha, a, b)) \leq \alpha$.*

# 5 Applications to Furstenberg Sets

In this section we will use the point-to-set principle, Theorem 5, in conjunction with the theorem of the previous section to give a new proof of a result by Molter and Rela on Furstenberg sets. A *set of Furstenberg type* with parameter $\alpha$ is a set $E \subseteq \mathbb{R}^2$ such that, for every $e \in S^1$, there is a line $\ell_e$ in the direction $e$ satisfying $\dim_H(E \cap \ell_e) \geq \alpha$. Finding the minimum possible dimension of such a set is an important open problem with connections to Falconer's distance set conjecture and to Kakeya sets [5, 20].

Molter and Rela [17] introduced a natural generalization of Furstenberg sets, in which the set of directions may itself have fractal dimension. Formally, a set $E \subseteq \mathbb{R}^2$ is in the class $F_{\alpha\beta}$ if there is some set $J \subseteq S^1$ such that $\dim_H(J) \geq \beta$ and for every $e \in J$, there is a line $\ell_e$ in the direction $e$ satisfying $\dim_H(E \cap \ell_e) \geq \alpha$. They proved the following lower bound on the dimension of such sets.

**Theorem 14.** (Molter and Rela [17]) *For all $\alpha, \beta \in (0,1]$ and every set $E \in F_{\alpha\beta}$,*

$$\dim_H(E) \geq \alpha + \frac{\beta}{2}.$$

We will now give a new proof of this theorem, using the theorems of the previous section.

*Proof of Theorem 14.* Let $\alpha, \beta \in (0,1]$, $\epsilon > 0$, and $E \in F_{\alpha\beta}$. Let $A \subseteq \mathbb{N}$ be an oracle testifying to the Hausdorff dimension of $E$; i.e.,

$$\dim_H(E) = \sup_{z \in E} \dim^A(z).$$

Let $e \in S^1$ satisfy $\dim^A(e) > \beta - \epsilon$. Note that such a direction exists by the point-to-set principle. Let $l_e$ be a line in direction $e$ such that $\dim_H(l_e \cap E) \geq \alpha$. Let $a, b \in \mathbb{R}$ be the reals such that $L_{a,b} = l_e$. Note that $\dim^A(a,b) = \dim^A(e)$ because the mapping $e \mapsto a$ is computable and bi-Lipschitz in a neighborhood of $e$. Let $S = \{x \mid (x, ax+b) \in E \cap l_e\}$. Note that this implies that $\dim_H(S) \geq \alpha$. We then have that

| | | |
|---|---|---|
| $\forall a, b$ | $1 \in \mathrm{sp}(L_{a,b})$ | |
| $\dim(a, b) = 2$ | $\mathrm{sp}(L_{a,b}) = [1, 2]$ | |
| $\dim(a, b) \geq 1$ | $[1, 2] \subseteq \mathrm{sp}(L_{a,b})$ | $\mathrm{sp}(L_{a,b}) \subseteq [1, 2]$ Except for a set of dimension at most $\frac{1}{2}$ |
| $\dim(a, b) = d < 1$ | $[2d, 1 + d] \subseteq \mathrm{sp}(L_{a,b})$, | $\mathrm{sp}(L_{a,b}) \subseteq [d, 1 + d]$ Except a set of dimension at most $\frac{d}{2}$ |
| $\dim(a, b) = 0$ | $\mathrm{sp}(L_{a,b}) = [0, 1]$ | |
| $\dim(a, b) = \mathrm{Dim}(a, b)$ | $[d, 1 + d\}] \subseteq \mathrm{sp}(L_{a,b})$, $d = \min\{1, \dim(a, b)\}$ | |

Table 2: Updated table of the dimension spectra of lines.

$$\dim_H(E) \geq \sup_{x \in S} \dim^A(x, ax + b).$$

Therefore, to complete the proof, it suffices to show that there exists a point $x \in S$ such that

$$\dim^A(x, ax + b) \geq \alpha + \frac{\beta}{2} - \epsilon. \tag{2}$$

By Theorem 4, relativized to $A$, the set of all $x$ such that $\dim^A(x, ax+b) \leq \alpha + \frac{\beta}{2}$ has Hausdorff dimension at most $\alpha - \epsilon$. Since $\dim_H(S) \geq \alpha$, this implies that there is a point $x \in S$ which satisfies (2), and the proof is complete. □

# 6 Conclusion and Future Directions

In this paper, we have given two new results on the dimension spectra of lines in the plane, summarized in Table 2. The first gives a partial answer to the question posed by Lutz, asking whether, for every line $L_{a,b}$, $\mathrm{sp}(L_{a,b})$ contains a unit interval. We showed that if $\dim(a, b) \geq 1$, then this is true. Together with a previous result of N. Lutz and Stull, this implies the following.

**Corollary 15.** *For every $a, b \in \mathbb{R}$, $\mathrm{sp}(L_{a,b})$ contains an interval.*

However we still do not have complete answer to Lutz's question. This is an important open problem, and one that seems to require new techniques to solve.

We have also given the first nontrivial lower bound on the dimension spectrum of a given line. An important open problem is to improve the bounds given here. This would be not only an intrinsically interesting result, but would likely give improved bounds on Furstenberg sets. Another interesting direction for future research is to construct lines with "many" points of small dimension. In particular, for a given $\alpha > 0$, is there a line $L_{a,b}$ such that

$$\dim_H(A(\alpha, a, b)) = \alpha?$$

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# A Technical Appendix

In this section we formally prove Lemma 8. In order to prove this, we will make use of the following lemma due to N. Lutz and Stull [13].

**Lemma A.1.** *Let $a, b, x \in \mathbb{R}$, $k \in \mathbb{N}$. Suppose that $r \in \mathbb{N}$, $\delta \in \mathbb{R}_+$, and $\varepsilon, \eta \in \mathbb{Q}_+$ satisfy the following conditions.*

1. $r \geq \log(2|a| + |x| + 6)$.
2. $K_r(a, b) \leq (\eta + \varepsilon)\, r$.
3. *For every $(u, v) \in \mathbb{R}^2$ such that $t = -\log \|(a, b) - (u, v)\| \leq r$ and $ux + v = ax + b$, $K_r(u, v) \geq (\eta - \varepsilon)\, r + \delta \cdot (r - t)$.*

*Then,*

$$K_r^A(a, b, x | x, ax + b) \leq \frac{4\varepsilon}{\delta} r + K(\varepsilon) + K(\eta) + O(\log r)\,.$$

We will also need the following pair of geometric facts.

**Observation A.2** (N. Lutz and Stull [13])**.** *Let $a, x, b \in \mathbb{R}$ and $r \in \mathbb{N}$. Let $(q_1, q_2) \in B_{2^{-r}}(x, ax + b)$.*

1. *If $(p_1, p_2) \in B_{2^{-r}}(a, b)$, then $|p_1 q_1 + p_2 - q_2| < 2^{-r}(|p_1| + |q_1| + 3)$.*
2. *If $|p_1 q_1 + p_2 - q_2| \leq 2^{-r}(|p_1| + |q_1| + 3)$, then there is some pair $(u, v) \in B_{2^{-r}(2|a|+|x|+5)}(p_1, p_2)$ such that $ax + b = ux + v$.*

**Lemma 8.** *Let $a, b, x \in \mathbb{R}$, $k \in \mathbb{N}$. Suppose that $r_1, \ldots, r_k \in \mathbb{N}$, $\delta \in \mathbb{R}_+$, and $\varepsilon, \eta_1, \ldots, \eta_k \in \mathbb{Q}_+$ satisfy the following conditions for every $1 \leq i \leq k$.*

1. $r_i \geq \log(2|a| + |x| + 6) + r_{i-1}$.
2. $K_{r_i}(a, b) \leq (\eta_i + \varepsilon)\, r_i$.
3. *For every $(u, v) \in \mathbb{R}^2$ such that $t = -\log \|(a, b) - (u, v)\| \in (r_{i-1}, r_i]$ and $ux + v = ax + b$, $K_{r_i}(u, v) \geq (\eta_i - \varepsilon)\, r_i + \delta \cdot (r_i - t)$.*

*Then for every oracle set $A \subseteq \mathbb{N}$,*

$$K_{r_k}^A(a, b, x \,|\, x, ax + b) \leq 2^k \left( K(\eta_1, \ldots, \eta_k) + K(\varepsilon) + \frac{4\varepsilon}{\delta} r_k + O(\log r_k) \right)\,.$$

*Proof.* Let $a, b, x \in \mathbb{R}$. We proceed by induction on $k$. By Corollary A.1, the conclusion holds for $k = 1$. Assume the conclusion holds for all $i < k$. Let $r_1, \ldots, r_k$, $\delta$, $\varepsilon$, $\eta_1, \ldots, \eta_k$, and $A$ be as described in the lemma statement.

Define an oracle Turing machine $M$ that does the following given oracle $A$ and input $\pi = \pi_1\pi_2\pi_3\pi_4\pi_5$ such that $U^A(\pi_1) = (q_1, q_2) \in \mathbb{Q}^2$, $U(\pi_2) = (s_1, \ldots, s_k) \in \mathbb{N}^k$, $U(\pi_3) = \zeta \in \mathbb{Q}$, $U(\pi_4) = \iota_1, \ldots, \iota_k \in \mathbb{Q}$ and $U^A(\pi_5, q_1, q_2) = h \in \mathbb{Q}^2$

For every program $\sigma \in \{0,1\}^*$ with $\ell(\sigma) \leq (\iota_k+\zeta)s_k$, in parallel, $M$ simulates $U(\sigma)$. If one of the simulations halts with some output $(p_1, p_2) \in \mathbb{Q}^2 \cap B_{2^{-r_{k-1}}}(h)$ such that

$$|p_1 q_1 + p_2 - q_2| < 2^{-s_2}(|p_1| + |q_1| + 3)\,,$$

then $M$ halts with output $(p_1, p_2, q_1)$. Let $c_M$ be a constant for the description of $M$.

Now let $\pi_1$, $\pi_2$, $\pi_3$, $\pi_4$, and $\pi_5$ testify to $K_r^A(x, ax+b)$, $K(r_1, \ldots, r_k)$, $K(\varepsilon)$, $K(\eta_1, \ldots, \eta_k)$, and $K_{r_{k-1}, r_k}(a, b \,|\, x, ax+b)$ respectively, and let $\pi = \pi_1\pi_2\pi_3\pi_4\pi_5$.

By condition 2, there is some $(\hat{p}_1, \hat{p}_2) \in B_{2^{-r_k}}(a, b)$ such that $K(\hat{p}_1, \hat{p}_2) \leq (\eta + \varepsilon)r_k$, meaning that there is some $\hat{\sigma} \in \{0,1\}^*$ with $\ell(\hat{\sigma}) \leq (\eta+\varepsilon)r_k$ and $U(\hat{\sigma}) = (\hat{p}_1, \hat{p}_2)$. By Observation A.2(1),

$$|\hat{p}_1 q_1 + \hat{p}_2 - q_2| < 2^{-r_k}(|\hat{p}_1| + |q_1| + 3)\,,$$

for every $(q_1, q_2) \in B_{2^{-r_k}}(x, ax+b)$, so $M$ is guaranteed to halt on input $\pi$.

Hence, let $(p_1, p_2, q_1) = M(\pi)$. By Observation A.2(2), there is some

$$(u, v) \in B_{2^{\gamma - r_k}}(p_1, p_2) \subseteq B_{2^{-r_{k-1}}}(a, b)$$

such that $ux + v = ax + b$, where $\gamma = \log(2|a| + |x| + 5)$. We have

$$\|(p_1, p_2) - (u, v)\| < 2^{\gamma - r_k}$$

and $|q_1 - x| < 2^{-r_k}$, so

$$(p_1, p_2, q_1) \in B_{2^{\gamma+1-r_k}}(u, v, x)\,.$$

It therefore follows that

$$\begin{aligned}
K^A_{r_k - \gamma - 1, r_k}(u, v, x \,|\, x, ax+b) &\leq K(p_1, p_2, q_1) \\
&\leq \ell(\pi_1\pi_2\pi_3\pi_4\pi_5) + c_M \\
&\leq \ell(\pi_5) + K(r_1, \ldots, r_k) + K(\varepsilon) + K(\eta_1, \ldots, \eta_k) + c_M \\
&= \ell(\pi_5) + K(\varepsilon) + K(\eta_1, \ldots, \eta_k) + O(\log r_k)\,.
\end{aligned}$$

Applying Lemma 7 yields

$$K^A_{r_k}(u, v, x \,|\, x, ax+b) \leq \ell(\pi_5) + K(\varepsilon) + K(\eta_1, \ldots, \eta_k) + O(\log r_k). \tag{3}$$

By our inductive hypothesis, we have that

$$\begin{aligned}
\ell(\pi_5) &= K_{r_{k-1}, r_k}(a, b \,|\, x, ax+b) \\
&= K_{r_{k-1}}(a, b \,|\, x, ax+b) + O(\log r_{k-1}) \\
&\leq 2^{k-1}\left(K(\varepsilon) + K(\eta_1, \ldots, \eta_k) + \frac{4\varepsilon}{\delta} r_{k-1} + O(\log r_{k-1})\right)\,.
\end{aligned} \tag{4}$$

To complete the proof, we bound $K^A_{r_k}(a, b, x \,|\, u, v, x)$. If $t > r_k$, then

$$K^A_{r_k}(a, b, x \,|\, u, v, x) \leq \log(r_k)\,.$$

Otherwise, when $t \le r_k$, by our construction of $M$ and Lemma 7,

$$\begin{aligned}(\eta+\varepsilon)r_k &\ge K(p_1,p_2)\\ &\ge K_{r_k-\gamma}(u,v)\\ &\ge K_{r_k}(u,v) - O(\log r_k)\,.\end{aligned}$$

Combining this with condition 3 in the lemma statement and simplifying yields

$$r_k - t \le \frac{2\varepsilon}{\delta} r_k + O(\log r_k)\,.$$

Therefore, by Lemma 7, we have

$$\begin{aligned}K_{r_k}(a,b,x \mid u,v,x) &\le 2(r_k - t) + O(\log r_k)\\ &\le \frac{4\varepsilon}{\delta} r_k + O(\log r_k)\,,\end{aligned} \tag{5}$$

for every $t \in \mathbb{N}$.

Combining inequalities (3), (4) and (5) gives

$$\begin{aligned}K_{r_k}(a,b,x \mid x, ax+b) &\le K_{r_k}(u,v,x \mid x, ax+b) + K_{r_k}(a,b,x \mid u,v,x)\\ &\le K_{r_k}(u,v,x \mid x, ax+b) + \frac{4\varepsilon}{\delta} r_k + O(\log r_k)\\ &\le \ell(\pi_5) + K(\varepsilon) + K(\eta_1,\ldots,\eta_k) + \frac{4\varepsilon}{\delta} r_k + O(\log r_k)\\ &\le 2^k\left(K(\varepsilon) + K(\eta_1,\ldots,\eta_k) + \frac{4\varepsilon}{\delta} r_k + O(\log r_k)\right)\,.\end{aligned}$$

□ □

## A.1 Proof of Lemma 11

**Lemma 11.** *Let $(a,b) \in \mathbb{R}^2$ such that $\dim(a,b) \ge 1$. Assume that, for every $\tau > 0$ and every $M \in \mathbb{N}$ there are infinitely many $R \in \mathbb{N}$ such that*

$$K_s(a,b) \le (1+\tau)s,$$

*for every natural number $s \in [R, MR]$. Then, for every real number $d \in (0,1]$, there is a point $x$ such that $\dim(x, ax+b) = 1+d$.*

*Proof.* Let $d \in (0,1]$. For every $n \in \mathbb{N}$, let $\tau_n = \frac{1}{n}$ and $M_n = \frac{2^n}{d}$. Let $R_1, R_2, \ldots$ be a sequence of natural numbers such that the following hold.

1. $2^{R_n} < R_{n+1}$.
2. For every $n$, $R_n$ satisfies the hypothesis for the choices of $\tau_n$ and $M_n$.

We now define a real number $x$ such that

$$\dim(x, ax+b) = 1+d. \tag{6}$$

Let $y \in \mathbb{R}$ be a real number that is random relative to $(a,b)$. That is, for every $r \in \mathbb{N}$,

$$K_r^{a,b}(y) \geq r - \log r.$$

For every $n \in \mathbb{N}$, define $h_n = \frac{(M_n - 1)R_n}{2}$. For every $r \in \mathbb{N}$, let

$$x[r] = \begin{cases} 0 & \text{if } \frac{r}{h_n} \in (d, 1] \text{ for some } n \in \mathbb{N} \\ y[r] & \text{otherwise} \end{cases}$$

where $x[r]$ is the $r$th bit of $x$. Define $x \in \mathbb{R}$ to be the real number with this binary expansion.

We first claim that the dimension of $(x, ax + b)$ is at most $1 + d$. For every $n \in \mathbb{N}$, by our construction of $x$ and choice of $y$,

$$\begin{aligned} K_{h_n}(x) &= K_{dh_n}(x) + O(\log h_n) \\ &= K_{dh_n}(y) + O(\log h_n) \\ &\leq dh_n + O(\log h_n). \end{aligned}$$

Therefore, by the above bound and Lemma 7,

$$\begin{aligned} \dim(x, ax+b) &= \liminf_{r\to\infty} \frac{K_r(x, ax+b)}{r} \\ &= \liminf_{r\to\infty} \frac{K_r(x) + K_r(ax+b \,|\, x) + O(\log r)}{r} \\ &\leq \liminf_{r\to\infty} \frac{K_r(x) + r + O(\log r)}{r} \\ &\leq \liminf_{n\to\infty} \frac{K_{h_n}(x) + h_n + O(\log h_n)}{h_n} \\ &= d + 1. \end{aligned}$$

To complete the proof, it suffices to show that, for every $\eta \in \mathbb{Q} \cap (0, 1)$ and $\varepsilon \in \mathbb{Q}_+$,

$$\dim(x, ax + b) \geq \eta + d - \varepsilon. \tag{7}$$

To that end, let $\eta \in \mathbb{Q} \cap (0, 1)$ and $\varepsilon \in \mathbb{Q}_+$. To prove inequality (7), we will partition $\mathbb{N}$ into intervals, and focus on the complexity of $(x, ax + b)$ at each precision $r$ in these intervals. For every $n \in \mathbb{N}$, let $I_n = (dh_n, dh_{n+1}]$.

Fix $n \in \mathbb{N}$, and let $m = \frac{1-d}{1-\eta}$. We will first consider $r \in (dh_n, mh_n]$. Let $k = \frac{r}{dh_n}$, and define $r_i = idh_n$ for every $1 \leq i \leq k$. It is important to note that $k$ is bounded by a constant depending only on $\eta$ and $d$. In particular, this implies that $o(r_k)$ is sublinear for all $r_i$. Let $D_r = D(r_1, \ldots, r_k, a, b, \eta)$ be the oracle defined in Lemma 10. We first note that, by our assumption of $(a, b)$ on the interval $[R_n, MR_n]$ and Lemma 7,

$$\begin{aligned} K_{r_i, r_{i-1}}(a, b \,|\, a, b) &= K_{r_i}(a, b) - K_{r_{i-1}}(a, b) - O(\log r_i) \\ &\geq r_i - o(r_i) - (1 + \frac{1}{n})r_{i-1} - O(\log r_i) \\ &= r_i - r_{i-1} - \frac{r_{i-1}}{n} - o(r_i), \end{aligned}$$

for all sufficiently large $r$. Since $k$ is bounded by a constant, for all sufficiently large $n$, we have

$$K_{r_i,r_{i-1}}(a,b \,|\, a,b) > \eta(r_i - r_{i-1}) - o(r_i).$$

Hence, by Lemma 10,

$$|K^{D_r}_{r_i}(a,b) - \eta r_i| < o(r_k), \tag{8}$$

for all $1 \leq i \leq k$.

We now show that the conditions of Lemma 8 are satisfied relative to $D_r$. Item 1 of Lemma 8 holds for all sufficiently large $r$. For item 2, by the construction of $D_r$, for every $1 \leq i \leq k$,

$$\begin{aligned}
K^{D_r}_{r_i}(a,b) &= \eta r_1 + \sum_{j=2}^{i} \min\{\eta(r_j - r_{j-1}), K_{r_j,r_{j-1}}(z \,|\, z)\} + O(\log r_k) \\
&\leq \eta r_1 + \sum_{j=2}^{i} \eta(r_j - r_{j-1} + O(\log r_k) \\
&\leq \eta r_i + O(\log r_k) \\
&\leq (\eta + \varepsilon) r_i,
\end{aligned}$$

for all sufficiently large $r$.

Let $\delta = 1 - \eta$. To see that item 3 of Lemma 8 is satisfied for $i = 1$, let $(u,v) \in B_1(a,b)$ such that $ux + v = ax + b$ and $t = -\log \|(a,b) - (u,v)\| \leq r_1$. Then, by Lemmas 9 and 10, and our construction of $x$,

$$\begin{aligned}
K^{D_r}_{r_1}(u,v) &\geq K^{D_r}_{t}(a,b) + K^{D_r}_{r_1-t,r_1}(x|a,b) - O(\log r_1) \\
&\geq \min\{\eta r_1, K_t(a,b)\} + K_{r_1-t}(x) - o(r_k) \\
&\geq \min\{\eta r_1, t - o(t)\} + (\eta + \delta)(r_1 - t) - o(r_k) \\
&\geq \min\{\eta r_1, \eta t - o(t)\} + (\eta + \delta)(r_1 - t) - o(r_k) \\
&\geq \eta t - o(t) + (\eta + \delta)(r_1 - t) - o(r_k)\,,
\end{aligned}$$

We conclude that $K^{D_r}_{r_1}(u,v) \geq (\eta - \varepsilon) r_1 + \delta(r_1 - t)$, for all sufficiently large $r$. To see that that item 3 is satisfied for $1 < i \leq k$, let $(u,v) \in B_{2^{-r_{i-1}}}(a,b)$ such that $ux + v = ax + b$ and $t = -\log \|(a,b) - (u,v)\| \leq r_i$. Since $(u,v) \in B_{2^{-r_{i-1}}}(a,b)$,

$$r_i - t \leq r_i - r_{i-1} = idh_j - (i-1)dh_j \leq dh_j + 1 \leq r_1 + 1\,.$$

Therefore, by Lemma 9, inequality (8), and our construction of $x$,

$$\begin{aligned}
K^{D_r}_{r_i}(u,v) &\geq K^{D_r}_{t}(a,b) + K^{D_r}_{r_i-t,r_i}(x|a,b) - O(\log r_i) \\
&\geq \min\{\eta r_i, K_t(a,b)\} + K_{r_i-t}(x) - o(r_i) \\
&\geq \min\{\eta r_i, t - o(t)\} + (\eta + \delta)(r_i - t) - o(r_i) \\
&\geq \min\{\eta r_i, \eta t - o(t)\} + (\eta + \delta)(r_i - t) - o(r_i) \\
&\geq \eta t - o(t) + (\eta + \delta)(r_i - t) - o(r_i)\,.
\end{aligned}$$

We conclude that $K^{D_r}_{r_i}(u,v) \geq (\eta-\varepsilon)r_i + \delta(r_i - t)$, for all sufficiently large $r$. Hence the conditions of Lemma 8 are satisfied. Therefore, by applying Lemma 8 and appealing to inequality (8),

$$\begin{aligned} K_r(x, ax+b) \geq & K^{D_r}_r(x, ax+b) \\ \geq & K_r(a,b,x) - 2^k\left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1-\eta}r_k + O(\log r_k)\right) \\ = & K_r(a,b) + K_r(x \mid a,b) \\ & - 2^k\left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1-\eta}r_k + O(\log r_k)\right) \\ \geq & dr + \eta r - 2^k\left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1-\eta}r_k + O(\log r_k)\right). \end{aligned}$$

To complete the proof, we give lower bounds of $K_r(x, ax+b)$ for every $r \in [mh_n, dh_{n+1})$. By Lemma 7 and our construction of $x$,

$$\begin{aligned} K_r(x) &= K_{r,h_n}(x \mid x) + K_{h_n}(x) - o(r) \\ &= r - h_n + dh_n - o(r) \\ &\geq \eta r - o(r)\,. \end{aligned}$$

The proof of Theorem 1 gives

$$\begin{aligned} K_r(x, ax+b) &\geq K_r(x) + \dim(x)r - o(r) \\ &\geq \eta r + dr - o(r) \\ &\geq r(d+\eta) - \varepsilon r. \end{aligned}$$

Putting together the lower bounds of $K_r(x, ax+b)$ on the intervals $(dh_n, mh_n)$ and $[mh_n, dh_{n+1}]$ shows that

$$\dim(x, ax+b) \geq 1 + d,$$

and the proof is complete. □

## A.2 Claims in Theorem 3

*Claim 1:* $\dim(x, ax+b) \leq 1 + d$.

*Proof.* For every $n \in \mathbb{N}$, by our construction of $x$ and choice of $y$,

$$\begin{aligned} K_{h_n}(x) &= K_{dh_n}(x) + O(\log h_n) \\ &= K_{dh_n}(y) + O(\log h_n) \\ &\leq dh_n + O(\log h_n). \end{aligned}$$

Therefore, by the above bound and Lemma 7,

$$\begin{aligned}\dim(x, ax+b) &= \liminf_{r\to\infty} \frac{K_r(x, ax+b)}{r}\\ &= \liminf_{r\to\infty} \frac{K_r(x) + K_r(ax+b\,|\,x) + O(\log r)}{r}\\ &\leq \liminf_{r\to\infty} \frac{K_r(x) + r + O(\log r)}{r}\\ &\leq \liminf_{n\to\infty} \frac{K_{h_n}(x) + h_n + O(\log h_n)}{h_n}\\ &= d+1.\end{aligned}$$

□

*Claim 2:* $K_{h_n}(x, ax+b) \geq dh_n + \eta h_n - 2^k \left(K(\varepsilon) + kK(\eta) + \frac{4\varepsilon}{1-\eta} h_n + O(\log h_n)\right)$.

*Proof.* Let $D_r = D(r_1, \ldots, r_k, a, b, \eta)$ be the oracle defined in Lemma 10. For every $1 \leq i < k$, define

$$\eta_i = \frac{\eta r_1 + \sum_{j=2}^{i} \min\{\eta(r_j - r_{j-1}), K_{r_j, r_{j-1}}(z\,|\,z)\}}{r_i},$$

and $\eta_k = \eta$. It is clear that for every $1 \leq i < k$,

$$K^{D_r}_{r_i}(a, b) = \eta_i r_i + O(\log r_k). \tag{9}$$

For $i = k$, note that, by our choice of $h_n$, we have

$$\begin{aligned}K_{h_n, r_{k-1}}(a, b\,|\,a, b) &\geq K_{h_n}(a, b) - K_{r_{k-1}}(a, b) - \log h_n\\ &\geq (\mathrm{Dim}(a, b) - \frac{1}{n})h_n - \mathrm{Dim}(a, b) r_{k-1} - o(h_n)\\ &= \mathrm{Dim}(a, b)(h_n - r_{k-1}) - \frac{h_n}{n} - o(h_n)\\ &> \eta(h_n - r_{k-1}) - \frac{h_n}{n} - o(h_n).\end{aligned}$$

Hence

$$|K^{D_r}_{h_n}(a, b) - \eta h_n| < \frac{h_n}{n} + o(h_n) \tag{10}$$

We will now show that the conditions of Lemma 8 hold for these choices, relative to $D_r$. By equation (9), for every $1 \leq i < k$,

$$\begin{aligned}K^{D_r}_{r_i}(a, b) &= \eta r_1 + \sum_{j=2}^{i} \min\{\eta(r_j - r_{j-1}), K_{r_j, r_{j-1}}(z\,|\,z)\} + \log r_k\\ &= \eta_i r_i + \log r_k\\ &\leq (\eta_i + \varepsilon) r_i,\end{aligned}$$

for sufficiently large $r$. For $i = k$, $K^{D_r}_{h_n}(a, b) \leq (\eta + \varepsilon)h_n$ holds immediately by equation (10). Therefore item 2 of Lemma 8 is satisfied.

Let $\delta = 1 - \eta$. To see that item 3 of Lemma 8 is satisfied for $i = 1$, let $(u, v) \in B_1(a, b)$ such that $ux + v = ax + b$ and $t = -\log \|(a, b) - (u, v)\| \leq r_1$. Then, by Lemmas 9, equation (9), and our construction of $x$,

$$\begin{aligned}
K^{D_r}_{r_1}(u, v) &\geq K^{D_r}_{t}(a, b) + K^{D_r}_{r_1-t,r_1}(x|a, b) - O(\log r_1)\\
&\geq \min\{\eta r_1, K_t(a, b)\} + K_{r_1-t}(x) - o(r_k)\\
&\geq \min\{\eta r_1, t - o(t)\} + (\eta + \delta)(r_1 - t) - o(r_k)\\
&\geq \min\{\eta r_1, \eta t - o(t)\} + (\eta + \delta)(r_1 - t) - o(r_k)\\
&\geq \eta t - o(t) + (\eta + \delta)(r_1 - t) - o(r_k)\,.
\end{aligned}$$

We conclude that $K^{D_r}_{r_1}(u, v) \geq (\eta - \varepsilon)r_1 + \delta(r_1 - t)$, for all sufficiently large $r$. To see that that item 3 is satisfied for $1 < i \leq k$, let $(u, v) \in B_{2^{-r_{i-1}}}(a, b)$ such that $ux + v = ax + b$ and $t = -\log \|(a, b) - (u, v)\| \leq r_i$. Since $(u, v) \in B_{2^{-r_{i-1}}}(a, b)$,

$$r_i - t \leq r_i - r_{i-1} = idh_j - (i - 1)dh_j \leq dh_j + 1 \leq r_1 + 1\,.$$

Therefore, by Lemma 9, Lemma 10, and our construction of $x$,

$$\begin{aligned}
K^{D_r}_{r_i}(u, v) &\geq K^{D_r}_{t}(a, b) + K^{D_r}_{r_i-t,r_i}(x|a, b) - O(\log r_i)\\
&\geq \min\{\eta_i r_i, K_t(a, b)\} + K_{r_i-t}(x) - o(r_i)\\
&\geq \min\{\eta_i r_i, t - o(t)\} + (\eta_i + \delta)(r_i - t) - o(r_i)\\
&\geq \min\{\eta_i r_i, \eta_i t - o(t)\} + (\eta_i + \delta)(r_i - t) - o(r_i)\\
&\geq \eta_i t - o(t) + (\eta_i + \delta)(r_i - t) - o(r_i)\,.
\end{aligned}$$

We conclude that $K^{D_r}_{r_i}(u, v) \geq (\eta - \varepsilon)r_i + \delta(r_i - t)$, for all sufficiently large $r$. Hence the conditions of Lemma 8 are satisfied. Therefore, by applying Lemma 8 and appealing to inequality (8),

$$\begin{aligned}
K_{h_n}(x, ax + b) \geq& K^{D_r}_{h_n}(x, ax + b)\\
\geq& K_{h_n}(a, b, x) - 2^k \left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{\delta}h_n + O(\log h_n)\right)\\
=& K_{h_n}(a, b) + K_{h_n}(x \mid a, b)\\
&- 2^k \left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1 - \eta}h_n + O(\log h_n)\right)\\
\geq& dh_n + \eta h_n - 2^k \left(K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1 - \eta}h_n + O(\log h_n)\right).
\end{aligned}$$

□

*Claim 3:* For all $r \in [dh_n, h_n)$,

$$K_r(x, ax + b) \geq r(d + \eta) - 2^k \left(K(\varepsilon) + K(\eta) + \tfrac{4\varepsilon}{1-\eta}h_n + O(\log h_n)\right).$$

*Proof.* By our construction of $x$, Lemma 6 and Claim 2,

$$\begin{aligned} K_r(x, ax+b) + r - h_n &\geq K_{h_n}(x, ax+b) \\ &\geq dh_n + \eta h_n - 2^k \left( K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1-\eta} h_n + O(\log h_n) \right). \end{aligned}$$

It is not difficult to verify that, for all $r < h_n$,

$$dh_n + \eta h_n - r + h_n > r(d + \eta).$$

Thus, we have that, for all $r \in (dh_n, h_n)$,

$$K_r(x, ax+b) \geq r(d+\eta) - 2^k \left( K(\varepsilon) + K(\eta) + \tfrac{4\varepsilon}{1-\eta} h_n + O(\log h_n) \right).$$

□

*Claim 4:* There is a precision $h_n < j \leq ch_n$ such that

$$K_j(x, ax+b) \geq j - (h_n - dh_n) + \eta j - 2^k \left( K(\varepsilon) + kK(\eta) + \tfrac{4\varepsilon}{1-\eta} j + O(\log j) \right).$$

*Proof.* Let $j \in \mathbb{N}$ be the first precision greater than $h_n$ such that

$$K_{j,h_n}(a, b \,|\, a, b) \geq \eta(j - h_n) \tag{11}$$

As discussed in the proof of Theorem 3, there is a constant $c$, depending only on $\tau$, so that such a $j$ must exist in $(h_n, ch_n]$. Let $k = \frac{j}{h_n}$, and define $r_i = ih_n$ for every $1 \leq i \leq k$. Let $D_r = D(r_1, \ldots, r_k, a, b, \eta)$ be the oracle defined in Lemma 10. The point of inequality (11) is that, when combined with the assumption that $j$ is the first precision for which it holds, it implies that

$$K_j^{D_r}(a, b) = \eta j + O(\log j).$$

This allows us, by essentially the same proof as that of Claim 2, to apply Lemma 8, yielding

$$\begin{aligned} K_j(x, ax+b) &\geq K_r^{D_r}(x, ax+b) \\ &\geq K_r^{D_r}(a, b, x) - 2^k \left( K(\varepsilon) + kK(\eta) + \frac{4\varepsilon}{1-\eta} j + O(\log j) \right) \\ &\geq j - (h_n - dh_n) + \eta j - 2^k \left( K(\varepsilon) + kK(\eta) + \frac{4\varepsilon}{1-\eta} j + O(\log j) \right), \end{aligned}$$

completing the proof of Claim 4. □

*Claim 5:* For every $r \in (h_n, j)$,

$$K_r(x, ax+b) \geq r(d+\eta) - cr(1-\eta) - 2^k \left( K(\varepsilon) + K(\eta) + \tfrac{4\varepsilon}{1-\eta} cr + O(\log cr) \right).$$

*Proof.* Applying

$$\begin{aligned}K_r(x, ax+b) &\geq K_j(x, ax+b) - 2(j-r) \\ &\geq j - (h_n - dh_n) + \eta j - 2^k \left( K(\varepsilon) + K(\eta) + \frac{4\varepsilon}{1-\eta} j + O(\log j) \right) - 2(j-r).\end{aligned}$$

It is not difficult to show that

$$K_r(x, ax+b) \geq r(d+\eta) - j(1-\eta) - 2^k \left( K(\varepsilon) + K(\eta) + \tfrac{4\varepsilon}{1-\eta} j + O(\log j) \right),$$

for all $r \in (h_n, j)$. Since $j \leq ch_n$, the claim follows. □

## A.3 Proofs for Section 4

**Lemma 13.** *Let $\alpha \in (0,1)$, $x, a, b \in \mathbb{R}$, and $n, r \in \mathbb{N}$ such that $2r^{-\frac{1}{2}} < \frac{1}{n}$. Assume that $K_r(x, ax+b) < \alpha r + \frac{K_r(a,b)}{2} - \frac{r}{n}$, and $K_r^{a,b}(x) \geq \alpha r$. Then, for every $y \in \mathbb{R}$, if $K_r(y, ay+b) < \alpha r + \frac{K_r(a,b)}{2}$, at least one of the following holds.*

1. *$t := -\log \|x - y\| \leq r^{\frac{1}{2}}$.*
2. *$K_r(y \mid a, b, x) < \alpha r$.*

*Proof.* Assume the hypothesis, but assume that neither condition is satisfied for some $y$. Then,

$$\begin{aligned}K_r(a, b, x, y) &\leq K_r(x, ax+b) + K_r(y, ay+b) + 2t \\ &< 2\alpha r + K_r(a,b) - \frac{r}{n} + 2t.\end{aligned}$$

However, by our hypothesis and Lemma 7 we have

$$\begin{aligned}K_r(a, b, x, y) &\geq K_r(a,b) + K_r(x \mid a, b) + K_r(y \mid a, b, x) - \log r \\ &\geq K_r(a,b) + 2\alpha r - \log r.\end{aligned}$$

Since condition (1) was assumed to not hold, we see that

$$\tfrac{1}{n} < 2r^{-\frac{1}{2}},$$

a contradiction. □

**Theorem 4.** *For every $a, b \in \mathbb{R}$ and $\alpha \in (0,1)$, $\dim_H(A(\alpha, a, b)) \leq \alpha$.*

*Proof.* Our goal is to show that $\dim_H(A(\alpha, a, b) \leq \alpha$. We will actually prove a stronger theorem. For every $n$, define

$$A_n(\alpha, a, b) = \{x \mid (\exists^\infty r)\, K_r(x, ax+b) < \alpha r + \tfrac{K_r(a,b)}{2} - \tfrac{r}{n}\}.$$

Note that to prove $\dim_H(A(\alpha, a, b) \leq \alpha$, it suffices to show that $\dim_H(A_n(\alpha, a, b)) \leq \alpha$ for every $n$. To see that $A(\alpha, a, b) \subseteq \cup_n A_n(\alpha, a, b)$, let $x \in A$. Then there is an $\epsilon > 0$ such that, for infinitely many $r$,

$$\begin{aligned} K_r(x, ax+b) &< \alpha r + \frac{\dim(a,b)}{2} r - \epsilon r \\ &< \alpha r + \frac{K_r(a,b) + g(r)}{2} - \epsilon r, \end{aligned}$$

where $g$ is a sublinear function. Therefore, for sufficiently large $n$ and $r$, $x \in A_n(\alpha, a, b)$. Since the Hausdorff dimension of a countable union of sets $\cup_n A_n$ is the supremum of $\dim_H(A_n)$, it suffices to show that, for every $n$, $\dim_H(A_n(\alpha, a, b)) \leq \alpha$.

Define the set

$$U = \{x \,|\, (\exists^\infty r)\, K_r^{a,b}(x) \leq \alpha r\}.$$

It is immediate that $\dim^{a,b}(x) \leq \alpha$, for all $x \in U$.

For every $r \in \mathbb{N}$, choose $x_r$ such that

$$\begin{aligned} &K_r^{a,b}(x) \geq \alpha r \\ &K_r(x, ax+b) < \alpha r + \frac{K_r(a,b)}{2} - \frac{r}{n}, \end{aligned}$$

if such an $x_r$ exists. To reduce the notational burden we will, without loss of generality, always assume that such an $x_r$ does exist. We then define

$$V = \{y \,|\, (\exists^\infty r)\, y \in (x_r - 2^{-r^{\frac{1}{2}}}, x_r + 2^{-r^{\frac{1}{2}}})\}.$$

Define oracle $R \subseteq \mathbb{N}$ which encodes the sequence $x_1, x_2, \ldots$ in the standard manner. Let $y \in V$, and let $r \in \mathbb{N}$ such that $y \in (x_r - 2^{-r^{\frac{1}{2}}}, x_r + 2^{-r^{\frac{1}{2}}})$. Then,

$$\begin{aligned} K^R_{r^{\frac{1}{2}}}(y) &\leq O(\log r) \\ &= O(\log r^{\frac{1}{2}}). \end{aligned}$$

Thus

$$\dim^R(y) = 0.$$

Let $x_1, x_2, \ldots$ be the sequence chosen above. Define

$$W = \{y \,|\, (\exists^\infty r)\, K_r(y \,|\, a, b, x_r) < \alpha r\}.$$

Let $y \in W$, and let $r \in \mathbb{N}$ such that $K_r(y \,|\, a, b, x_r) < \alpha r$. Then we have

$$\begin{aligned} K_r^{R,a,b}(y) &\leq K_r(y \,|\, a, b, x_r) + O(\log r) \\ &< \alpha r + O(\log r). \end{aligned}$$

Thus

$$\dim^{R,a,b}(y) \leq \alpha.$$

We now show that $A_n(\alpha, a, b) \subseteq U \cup V \cup W$. Let $y \in A_n(\alpha, a, b)$, and assume that $y \notin U \cup V$. So then $y$ has the following properties.

1. For infinitely many $r$, $K_r(y, ay + b) < \alpha r + \frac{K_r(a,b)}{2} - \frac{r}{n}$
2. For almost every $r$, $K_r^{a,b}(y) > \alpha r$.
3. For almost every $r$, $y \notin (x_r - 2^{-r^{\frac{1}{2}}}, x_r + 2^{-r^{\frac{1}{2}}})$.

Let $r \in \mathbb{N}$ be a sufficiently large integer such that item (1) holds. Then by Lemma 13, we must have that

$$K_r(y \mid a, b, x_r) < \alpha r.$$

Therefore, $y \in W$, and $A_n(\alpha, a, b) \subseteq U \cup V \cup W$. Hence $\dim^{R,a,b}(y) \leq \alpha$, and the proof is complete. $\square$